Thermodynamics with Strings & Walls - A Brandon Tribute!

CURVATURE CORRECTIONS TO DYNAMICS OF DOMAIN WALLS

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hep-th/9410095

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4-7-22

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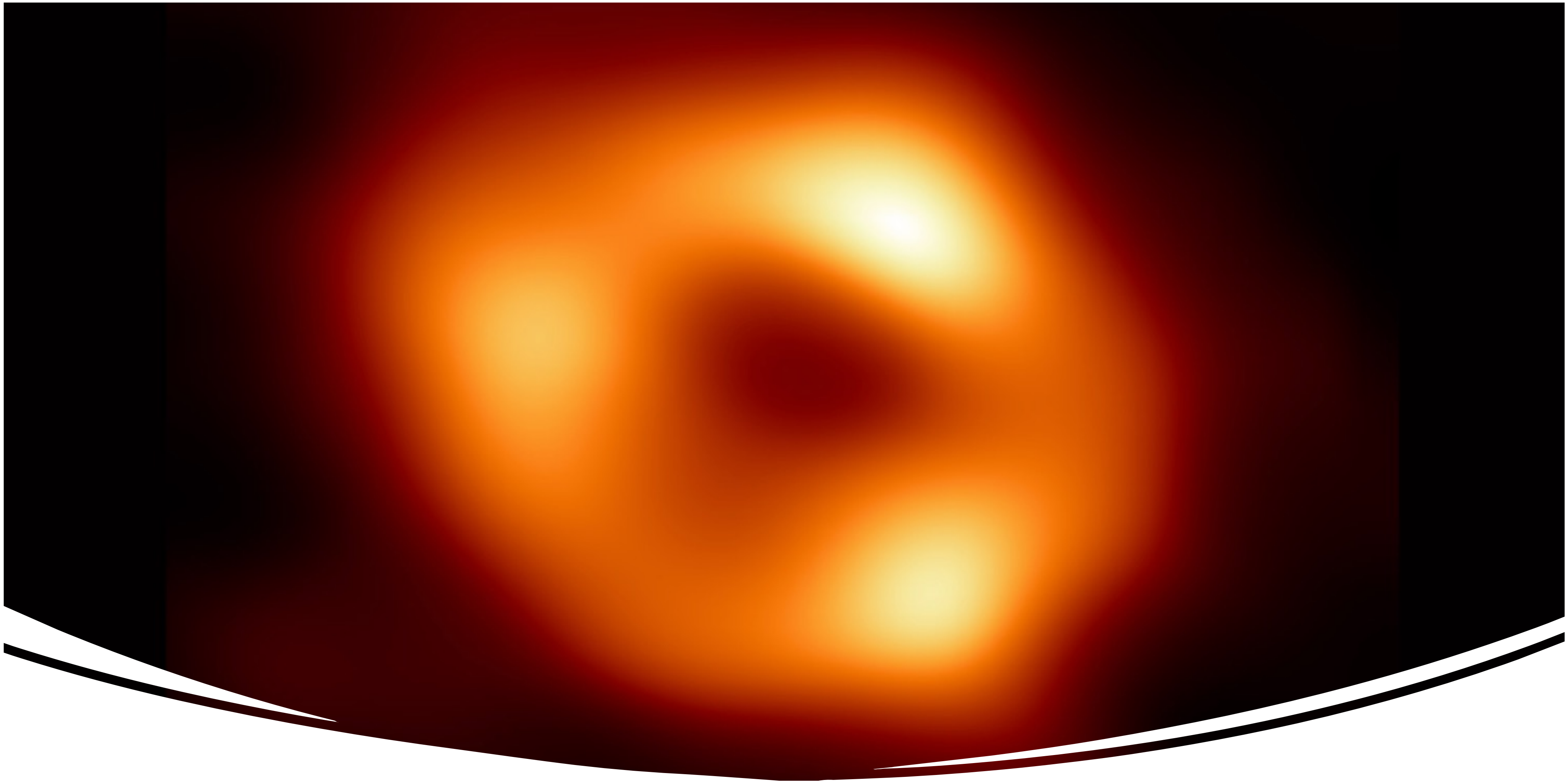


aCceleration
(w strings & walls)

- C in 4 – how to do thermodynamics
- C in 3 – how to do needlework!

Strings!

Acceleration in 4D

An accelerating black hole in 4D is described by the C-metric

$$ds^2 = \Omega^{-2}\left[f(r)dt^2 - \frac{dr^2}{f(r)} - r^2\left(\frac{d\theta^2}{g(\theta)} + g(\theta)\sin^2\theta\frac{d\phi^2}{K^2}\right)\right]$$

Where

$$f = \left(1 - \frac{2m}{r}\right)\left(1 - A^2r^2\right) + \frac{r^2}{\ell^2}$$

$$g = 1 + 2mA\cos\theta$$

$$\Omega = 1 + Ar\cos\theta$$

f determines horizon structure – black hole / acceleration / cosmological constant

Hong & Teo, CQG20 3629 (2003)

Conical Deficits

Simplest illustration of thermodynamics is with a simple deficit. We can "cut out" a portion of the angular coordinate – this is a legitimate GR solution and can be sourced by a cosmic string (more or less!)

$$ds^2_{\theta,\phi} \propto d\theta^2 + \frac{\theta^2}{K^2} d\phi^2$$

K measures a deficit or "cosmic string" on axis, keeping the angular coordinate at fixed periodicity

$$\delta = 2\pi \left(1 - \frac{1}{K}\right) = 8\pi\mu$$

c.f. Aryal, Ford, Vilenkin: PRD 34, 2263 (1986), Achucarro, Gregory, Kuijken: PRD 52 5729 (1995)

Thermodynamics, with strings attached!

Take Schwarzschild-AdS with a deficit: $f = 1-2m/r+r^2/\ell^2$

Black hole horizon defined by f=0, look at small changes in f.

Horizon still defined by f(r) = 0.

$$f(r_+ + \delta r_+) = f'(r_+)\delta r_+ + \frac{\partial f}{\partial m}\delta m + \frac{\partial f}{\partial \ell}\delta \ell = 0$$

Changes r+,
hence S

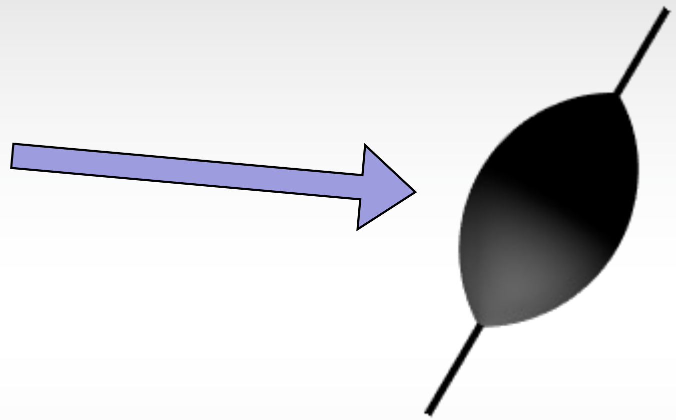
Changes m,
hence M

Changes ℓ,
hence Λ

Black Hole Thermodynamics

Temperature has usual definition, but entropy depends on K:

$$T = \frac{f'(r_+)}{4\pi} \qquad S = \frac{\pi r_+^2}{K}$$



-and we want to do thermodynamics including the string, so we have to take into account varying K.

$$\delta S = \frac{\pi r_+ \delta r_+}{K} - \pi r_+^2 \frac{\delta K}{K^2}$$

(S): Herdeiro, Kleihaus,Kunz,Radu: 0912:3386 [gr-qc]

Changing Tension

Tension is related to K:

$$\mu = \frac{1}{4}\left(1 - \frac{1}{K}\right)$$

So easily get

$$\delta\mu = \frac{\delta K}{4K^2}$$

Finally

$$P = -\Lambda = \frac{3}{8\pi\ell^2} \qquad V = \frac{4\pi r_+^3}{3K}$$

First Law with Tension

Putting together:

$$0 = \frac{2K}{r_+}\left(T\delta S + 2(m - r_+)\delta\mu + V\delta P - \delta(\frac{m}{K})\right)$$

So identify

$$M = \frac{m}{K}$$

Then also get Smarr relation:

$$M = 2TS - 2PV$$

Appels, RG, Kubiznak: 1702.00490 [hep-th]

Thermodynamic Length

The term multiplying the variation in tension is a "thermodynamic length"

$$\lambda = r_+ - m$$

Reinforces interpretation of M as **enthalpy**, if black hole grows, it swallows some string, but has also displaced the same amount of energy from environment.

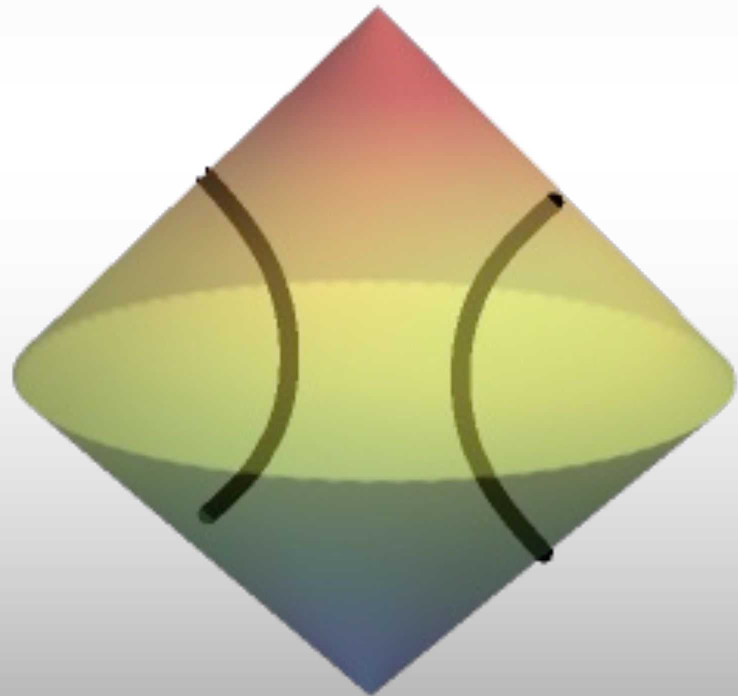
Kastor & Traschen: 1207:5415 [hep-th]

Acceleration

To get acceleration (A) add different tensions for N and S strings:

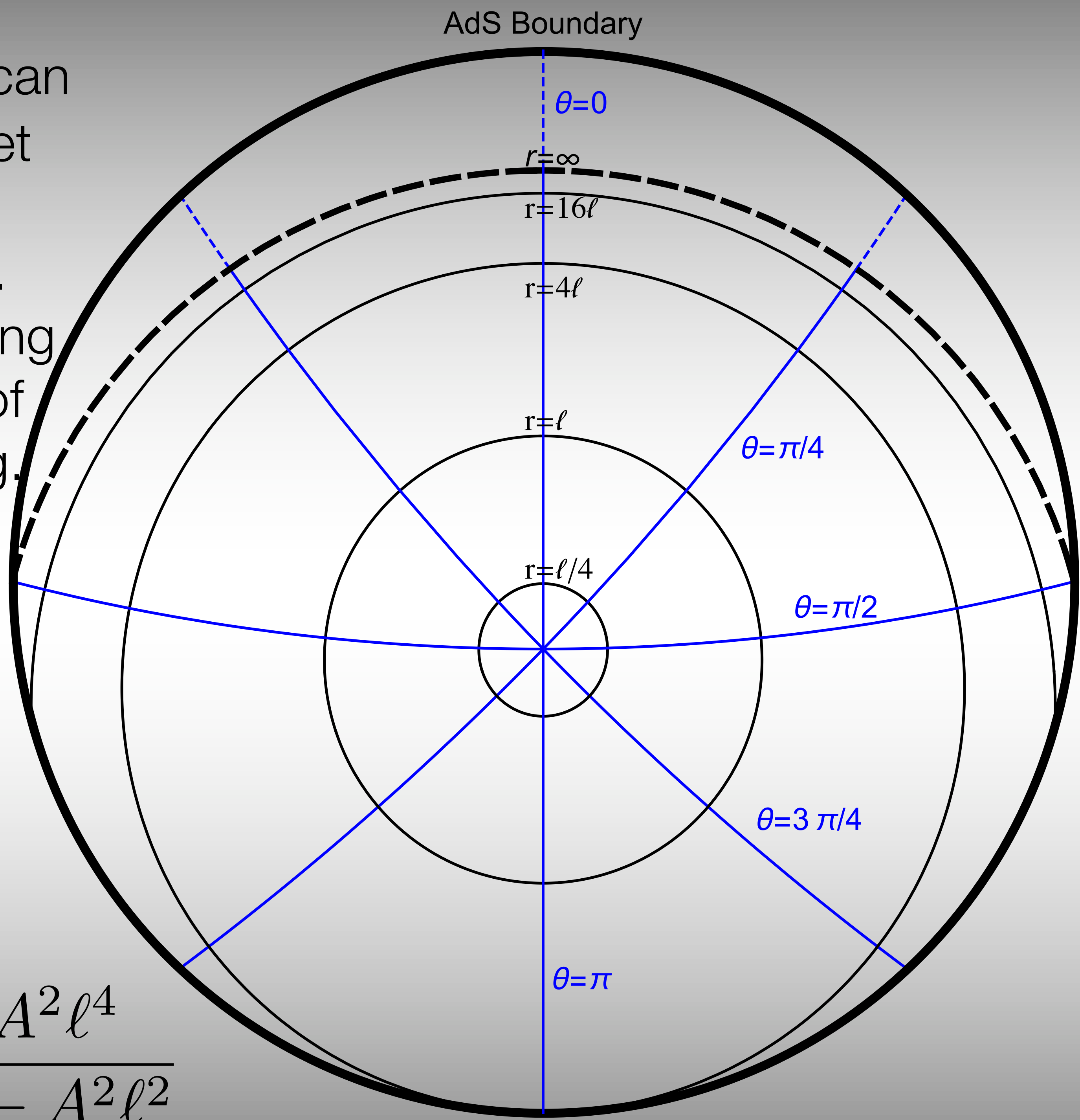
$$\delta_{\pm} = 2\pi\left(1 - \frac{g(0)}{K}\right) = 2\pi\left(1 - \frac{1 \pm 2mA}{K}\right) = \text{“}8\pi\mu_{\pm}\text{”}$$

In general, the C-metric has acceleration horizon, so thermodynamics would refer locally to black hole horizon – here we can see the nontrivial nature of the spacetime more readily.



We looked at slowly accelerating black holes to have only 1 horizon.

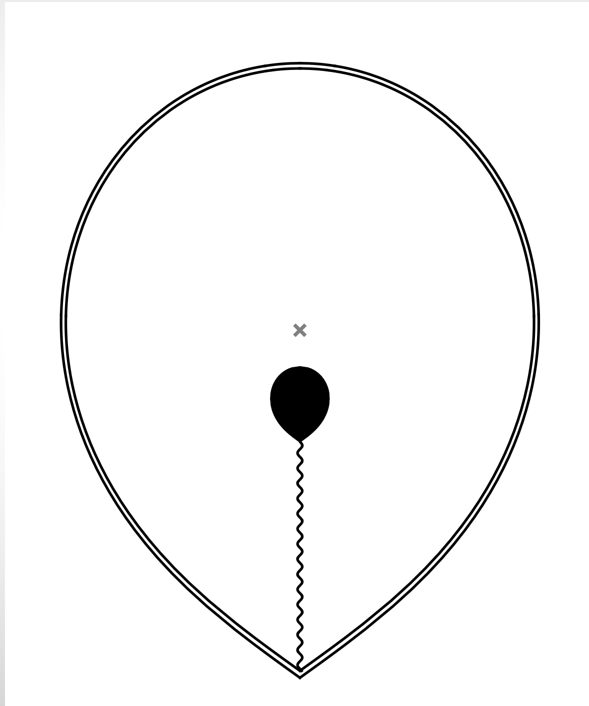
If m=0 and A**ℓ** <1, can change cords to get AdS from an off-centre perspective. An observer hovering away from centre of AdS is accelerating. This is SLOWLY ACCELERATING RINDLER



$$r = 0 \leftrightarrow R = \frac{A^2\ell^4}{1 - A^2\ell^2}$$

The Slowly Accelerating Black Hole

The slowly accelerating black hole in AdS is displaced from centre. It has a conical deficit running from the horizon to the boundary. The string tension provides the force that hold the black hole off-centre.



Technicalities

Finding the thermodynamics is via a similar method, looking at differences due to adding mass / angular momentum etc. Main technical ingredient is to note that "t" in the metric may not be the asymptotic time

$$ds^2 = \frac{1}{H^2}\left\{ \frac{f(r)}{\Sigma}\left[\frac{dt}{\alpha} - a\sin^2\theta \frac{d\varphi}{K}\right]^2 - \frac{\Sigma}{f(r)}dr^2 - \frac{\Sigma r^2}{h(\theta)}d\theta^2 - \frac{h(\theta)\sin^2\theta}{\Sigma r^2}\left[\frac{a dt}{\alpha} - (r^2+a^2)\frac{d\varphi}{K}\right]^2 \right\}$$

This will rescale temperature, and also changes computations of the mass.

$$f(r) = (1 - A^2 r^2)\left[1 - \frac{2m}{r} + \frac{a^2+e^2}{r^2}\right] + \frac{r^2+a^2}{\ell^2},$$

$$h(\theta) = 1 + 2mA\cos\theta + \left[A^2(a^2+e^2) - \frac{a^2}{\ell^2}\right]\cos^2\theta,$$

$$\Sigma = 1 + \frac{a^2}{r^2}\cos^2\theta, \qquad H = 1 + Ar\cos\theta.$$

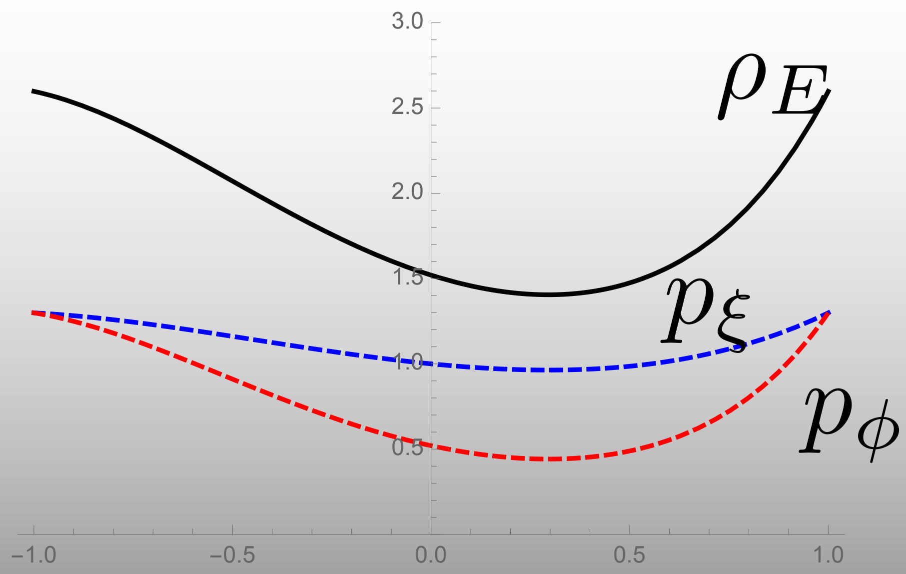
Thermodynamic M

We computed the mass via conformal completion as well as holographically (Fefferman-Graham):

$$\langle \mathcal{T}^{\mu}_{\nu} \rangle = \mathrm{diag}\left\{\rho_E, -\rho_E/2 + \Pi, \rho_E/2 - \Pi\right\}$$

$$\rho_E = \frac{m}{\alpha}(1 - A^2\ell^2 g)^{3/2}(2 - 3A^2\ell^2 g)$$

$$\Pi = \frac{3A^2\ell^2 gm}{2\alpha}(1 - A^2\ell^2 g)^{3/2}$$



Accelerating Thermodynamics

Integrate up the boundary stress-energy to get the mass:

$$M = \int \rho_E \sqrt{\gamma} = \frac{\alpha m}{K}$$

What is alpha? Setting m to zero, and demanding that the boundary is a round 2-sphere gives

$$\alpha = \sqrt{1 - A^2 \ell^2}$$

Get a consistent first law with corrections to V and TD length, and – can generalise to rotation

General Thermo Parameters

$$M = \frac{m(\Xi + a^2/\ell^2)(1 - A^2\ell^2\Xi)}{K\Xi\alpha(1 + a^2A^2)}$$

$$T = \frac{f'_+ r_+^2}{4\pi\alpha(r_+^2 + a^2)}\,, \quad S = \frac{\pi(r_+^2 + a^2)}{K(1 - A^2r_+^2)}\,,$$

$$Q = \frac{e}{K}\,, \quad \Phi = \Phi_t = \frac{er_+}{(r_+^2 + a^2)\alpha}\,,$$

$$J = \frac{ma}{K^2}\,, \quad \Omega = \Omega_H - \Omega_\infty\,, \quad \Omega_H = \frac{Ka}{\alpha(r_+^2 + a^2)}$$

$$P = \frac{3}{8\pi\ell^2}\,, \quad V = \frac{4\pi}{3K\alpha}\left[\frac{r_+(r_+^2 + a^2)}{(1 - A^2r_+^2)} + \frac{m[a^2(1 - A^2\ell^2\Xi) + A^2\ell^4\Xi(\Xi + a^2/\ell^2)]}{(1 + a^2A^2)\Xi}\right]$$

$$\lambda_\pm = \frac{r_+}{\alpha(1 \pm Ar_+)} - \frac{m}{\alpha}\frac{[\Xi + a^2/\ell^2 + \frac{a^2}{\ell^2}(1 - A^2\ell^2\Xi)]}{(1 + a^2A^2)\Xi^2} \mp \frac{A\ell^2(\Xi + a^2/\ell^2)}{\alpha(1 + a^2A^2)}$$

$$\Xi = 1 - \frac{a^2}{l^2} + A^2(e^2 + a^2) \qquad \alpha = \frac{\sqrt{(\Xi + a^2/\ell^2)(1 - A^2\ell^2\Xi)}}{1 + a^2A^2}$$

Chemical Expressions

Useful to express chemical potentials in terms of charges, rather than geometrical quantities, eg Christodoulou-Ruffini mass formula:

$$M^2(S,Q,J,P) = \frac{S}{4\pi}\left[\left(1+\frac{\pi Q^2}{S}+\frac{8PS}{3}\right)^2+\left(1+\frac{8PS}{3}\right)\frac{4\pi^2 J^2}{S^2}\right]$$

Physically, it is more natural to think in terms of average and differential deficits. We therefore encode:

$$\Delta = 1-2(\mu_+ + \mu_-) = \frac{\Xi}{K}$$

$$C = \frac{(\mu_- - \mu_+)}{\Delta} = \frac{mA}{K\Delta} = \frac{mA}{\Xi}$$

$$\left(\Xi = 1 + e^2A^2 - \frac{a^2}{\ell^2}(1-A^2\ell^2)\right)$$

$$M^2 = \frac{\Delta S}{4\pi}\left[\left(1+\frac{\pi Q^2}{\Delta S}+\frac{8PS}{3\Delta}\right)^2+\left(1+\frac{8PS}{3\Delta}\right)\left(\frac{4\pi^2 J^2}{(\Delta S)^2}-\frac{3C^2\Delta}{2PS}\right)\right]$$

$$V = \frac{2S^2}{3\pi M}\left[\left(1+\frac{\pi Q^2}{\Delta S}+\frac{8PS}{3\Delta}\right)+\frac{2\pi^2 J^2}{(\Delta S)^2}+\frac{9C^2\Delta^2}{32P^2S^2}\right],$$

$$T = \frac{\Delta}{8\pi M}\left[\left(1+\frac{\pi Q^2}{\Delta S}+\frac{8PS}{3\Delta}\right)\left(1-\frac{\pi Q^2}{\Delta S}+\frac{8PS}{\Delta}\right)-\frac{4\pi^2 J^2}{(\Delta S)^2}-4C^2\right],$$

$$\Omega = \frac{\pi J}{SM\Delta}\left(1+\frac{8PS}{3\Delta}\right),$$

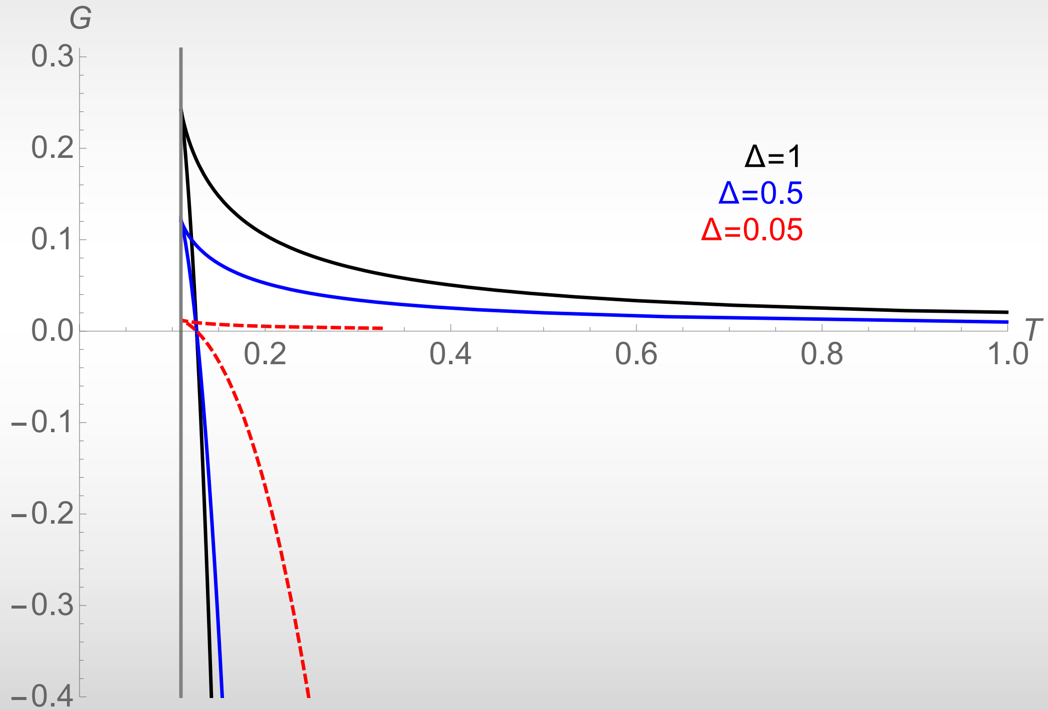
$$\Phi = \frac{Q}{2M}\left(1+\frac{\pi Q^2}{S\Delta}+\frac{8PS}{3\Delta}\right),$$

$$\lambda_\pm = \frac{S}{4\pi M}\left[\left(\frac{8PS}{3\Delta}+\frac{\pi Q^2}{\Delta S}\right)^2+\frac{4\pi^2 J^2}{(\Delta S)^2}\left(1+\frac{16PS}{3\Delta}\right)-(1\mp 2C)^2\pm\frac{3C\Delta}{2PS}\right]$$

RG & Scoins, 1904.09660

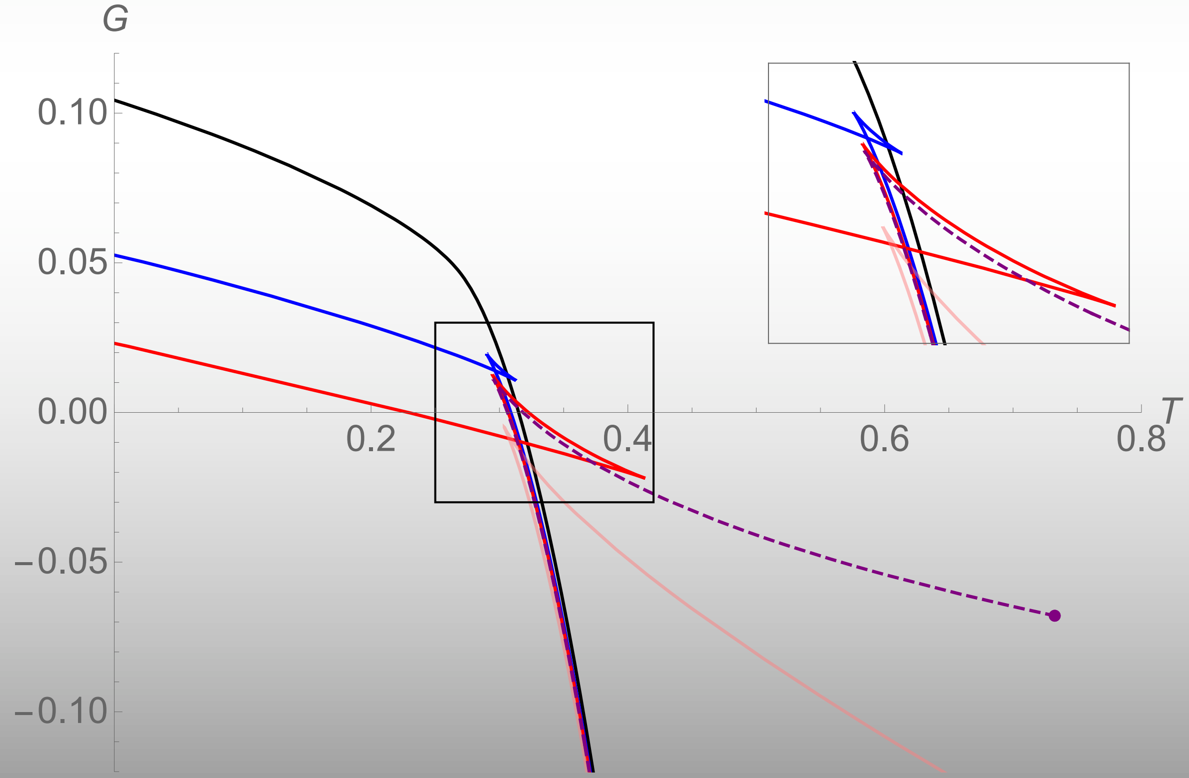
Accelerating Chemistry

These chemical expressions allow us to easily explore Hawking-Page transitions



Snapped Swallowtails

As well as analytic results for snapped swallowtails. (With Q or J, T can have two turning points in S at low pressures) Gradually dropping charge causes a swallowtail to appear in the free energy, which then snaps at a critical Q (J).



Abbasvandi et al. 1812.00384

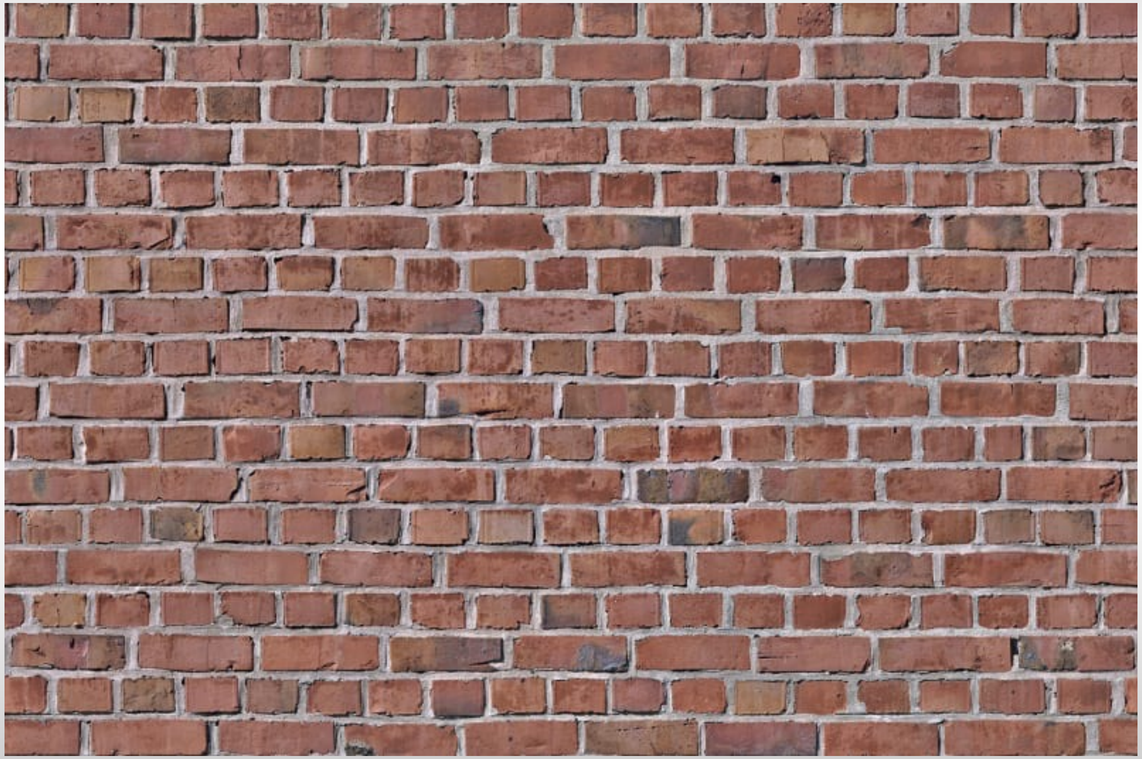
RG & Scoins

New Reverse Isoperimetric Inequality

Also proved a new reverse isoperimetric inequality appropriate for conical deficit black holes:

$$\left(\frac{3V}{4\pi}\right)^2 \geq \frac{1}{\Delta}\left(\frac{\mathcal{A}}{4\pi}\right)^3$$

WALLS!



"C" IN 3

We can look for an exact solution in 3D with the same type of structure:

$$ds^2 = \frac{1}{A^2(x-y)^2}\left[P(y)d\tau^2 - \frac{dy^2}{P(y)} - \frac{dx^2}{Q(x)}\right]$$

With general solution: $Q(x) = c + bx + ax^2\,,\qquad P(y) = \frac{1}{A^2\ell^2} - Q(y)$

Which, after coordinate rescaling/shifts reduces to:

Class	$Q(x)$	$P(y)$	Maximal range of x
I	$1-x^2$	$\frac{1}{A^2\ell^2} + (y^2-1)$	$\lvert x\rvert < 1$
II	x^2-1	$\frac{1}{A^2\ell^2} + (1-y^2)$	$x > 1$ or $x < -1$
III	$1+x^2$	$\frac{1}{A^2\ell^2} - (1+y^2)$	$\mathbb{R}$

Arenas-Henriquez, RG, Scoins: 2202:08823; Anber 0809:2789; Astorino 1101.2616

Accelerating Particle

Take each in turn. The first class looks very similar to the 4D C-metric (r=-1/Ay, t = ατ/A, x = cos(φ/K))

$$ds^2 = \frac{1}{[1 + Ar\cos(\phi/K)]^2}\left[f(r)\frac{dt^2}{\alpha^2} - \frac{dr^2}{f(r)} - r^2\frac{d\phi^2}{K^2}\right]$$

$$f(r) = 1 + (1 - A^2\ell^2)\frac{r^2}{\ell^2}$$

Slow Acceleration $A\ell < 1$ No horizon

Rapid Acceleration $A\ell > 1$ Acc. horizon

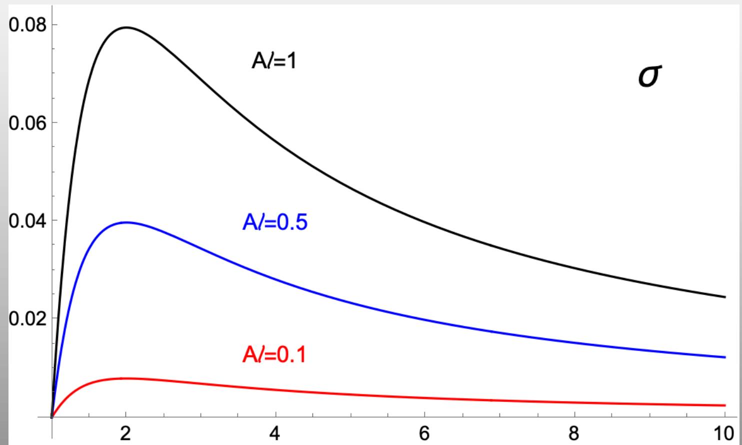
Slow Acceleration

The presence of K now indicates both a conical deficit (the particle) and a *domain wall* at $\phi=\pm\pi$, i.e. codimension 1 defect. The conical deficit at r=0 has a natural mass:

$$m_c = \frac{1}{4}\left(1 - \frac{1}{K}\right)$$

Because of the nonzero extrinsic curvature along $\phi=\pm\pi$, (thanks to A) there is a wall of tension

$$\sigma = \frac{A}{4\pi}\sin\left(\frac{\pi}{K}\right)$$

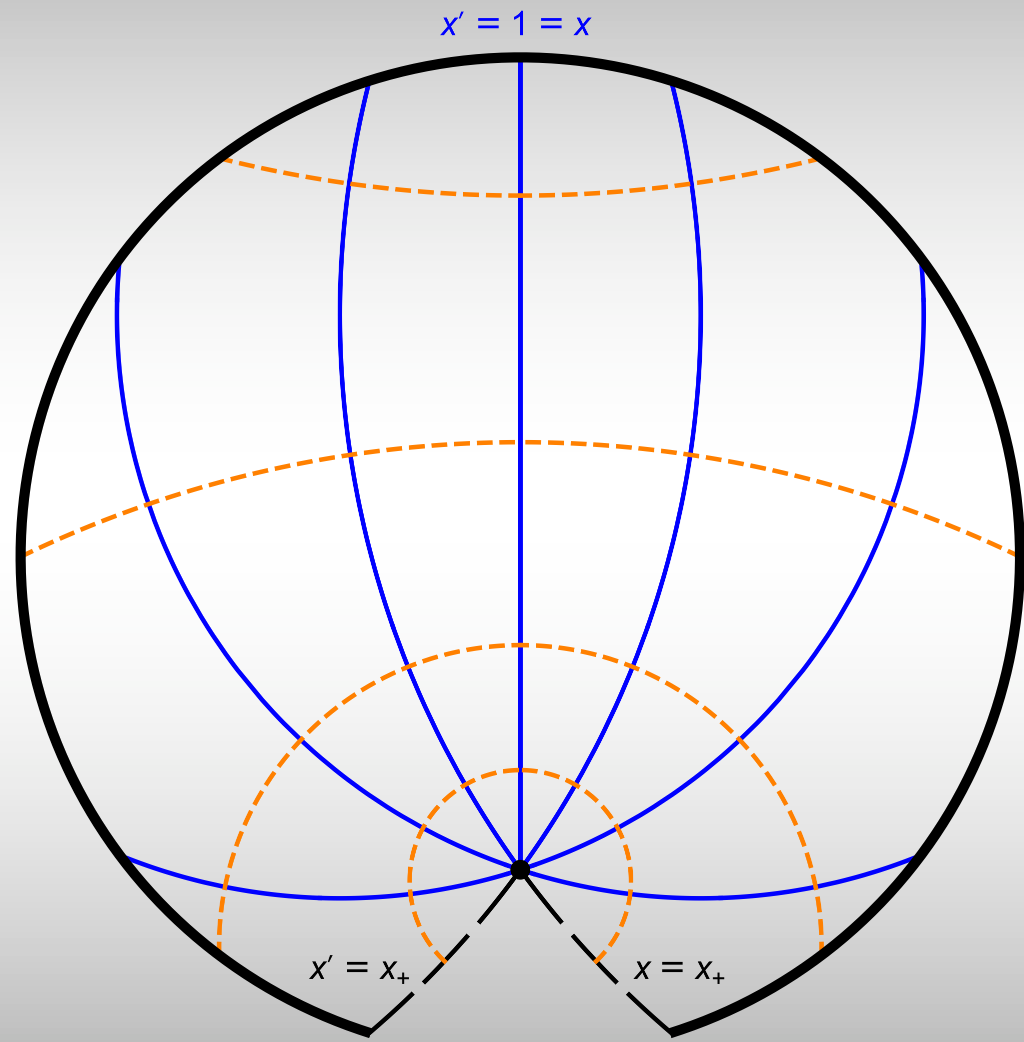


Slow Acceleration

We can do the same coord transformation as in the 4D slow acceleration case, to get the same sort of picture:

$$R_0 = \frac{A\ell^2}{\alpha}$$

A determines the displacement from origin, and x_+ both the particle "mass" and wall tension.



Particle Mass?

Can follow the same Fefferman-Graham prescription as for 4D, giving the expected boundary metric:

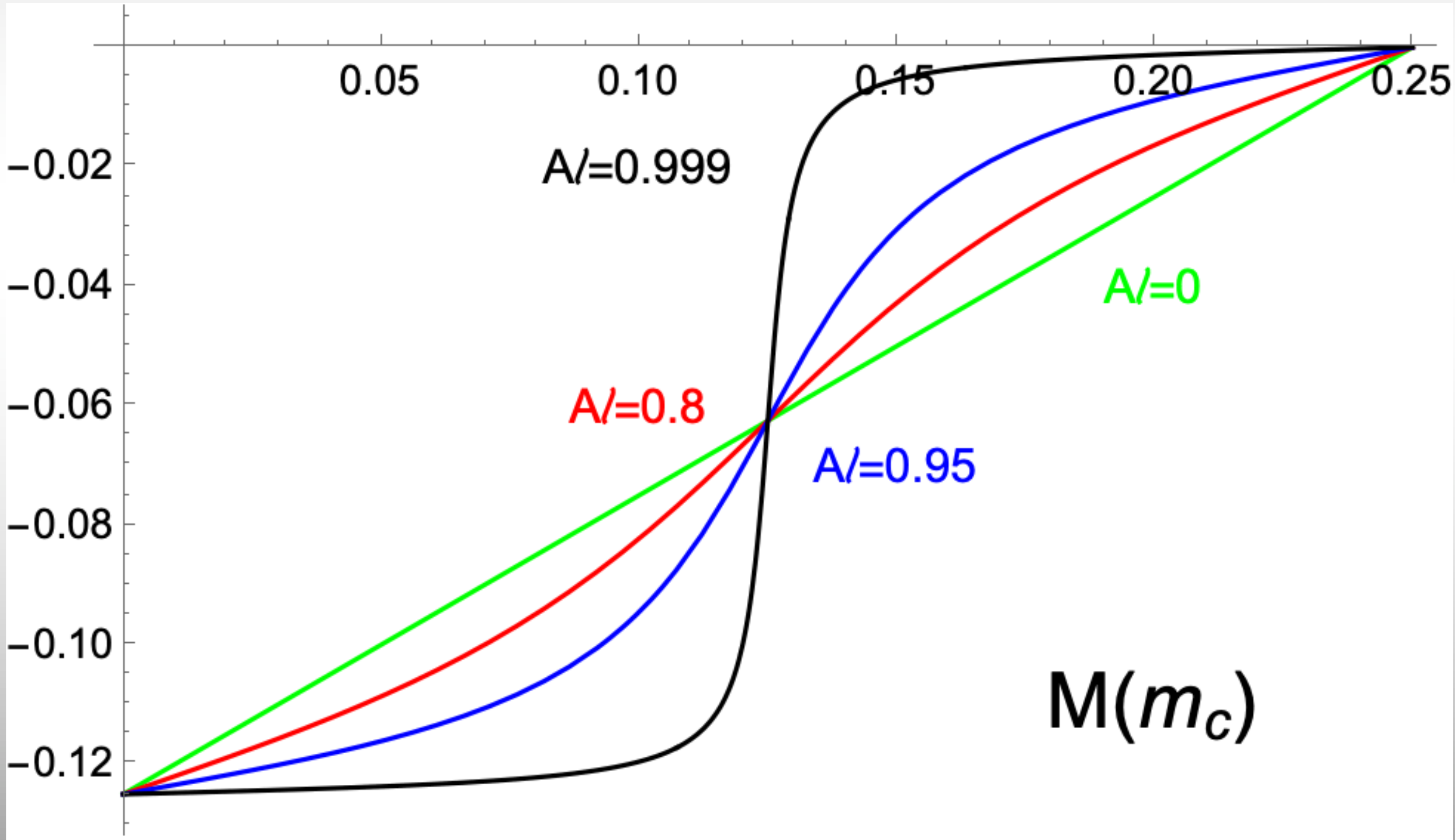
$$\gamma_0 = \frac{\omega(\xi)^2}{A^2}\left[d\tau^2 - A^2\ell^2\frac{d\xi^2}{1-\xi^2}\right]$$

And, after setting alpha to the same value as 4D, the mass:

$$M = -\frac{1}{8\pi}\left(\frac{\pi}{2} - \arctan\left[\frac{\cot\left(\frac{\pi}{K}\right)}{\sqrt{1-A^2\ell^2}}\right]\right)$$

Holographic Mass

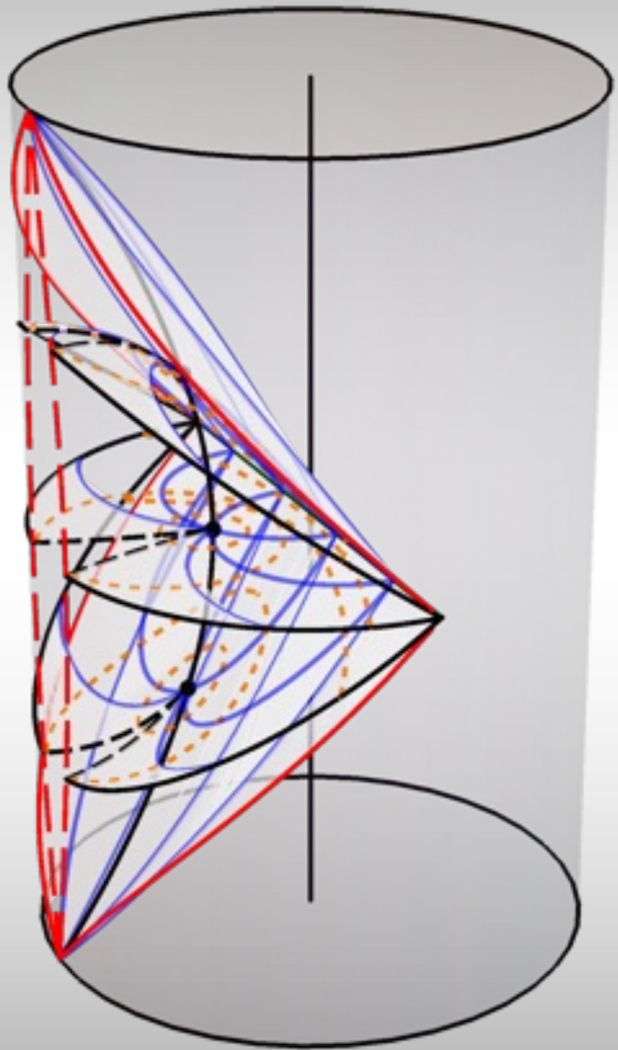
Compare to "particle" mass from conical deficit:

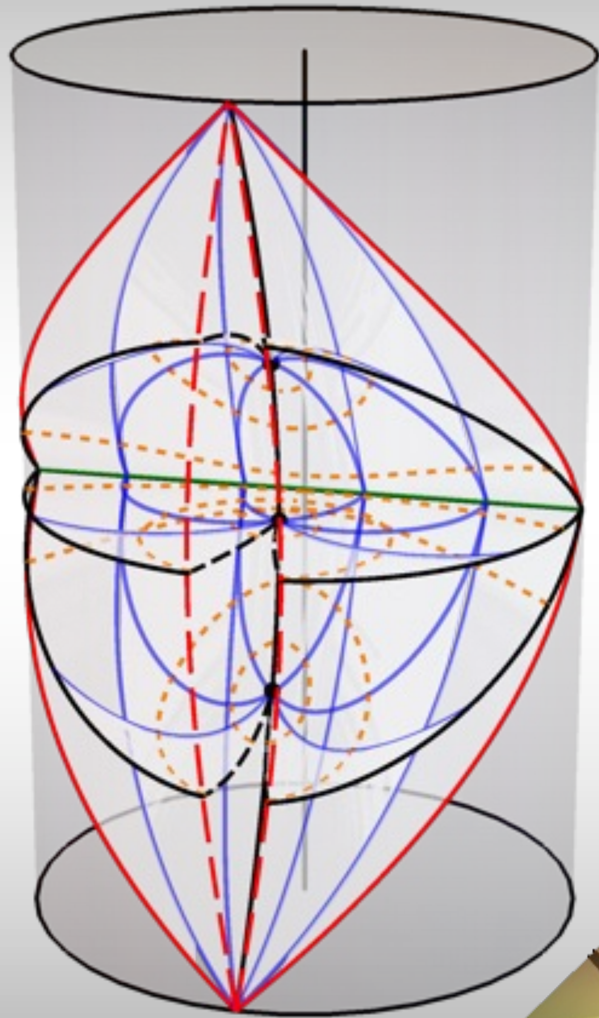


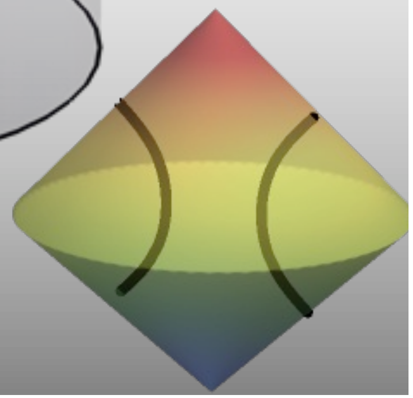
New Solutions?

Although these have been derived as “new” solutions, we know that in 3D, gravity does not propagate, so any ”vacuum” solution has to be locally equivalent to AdS. The transformation formulae for the various solutions are quite lengthy, but give an interesting alternative viewpoint, and help with understanding the “BTZ” family of solutions.

Rapidly accelerating light Particle

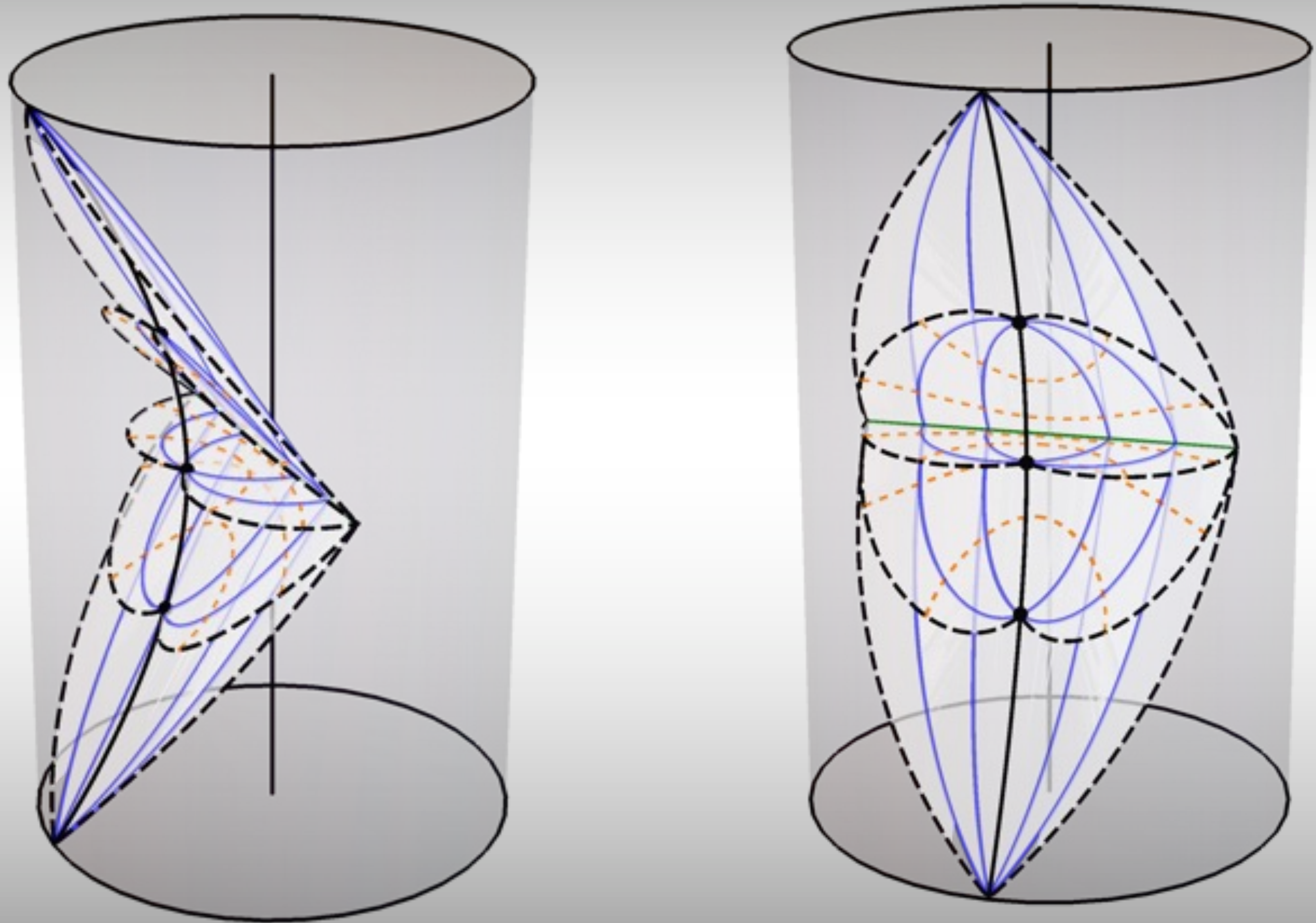




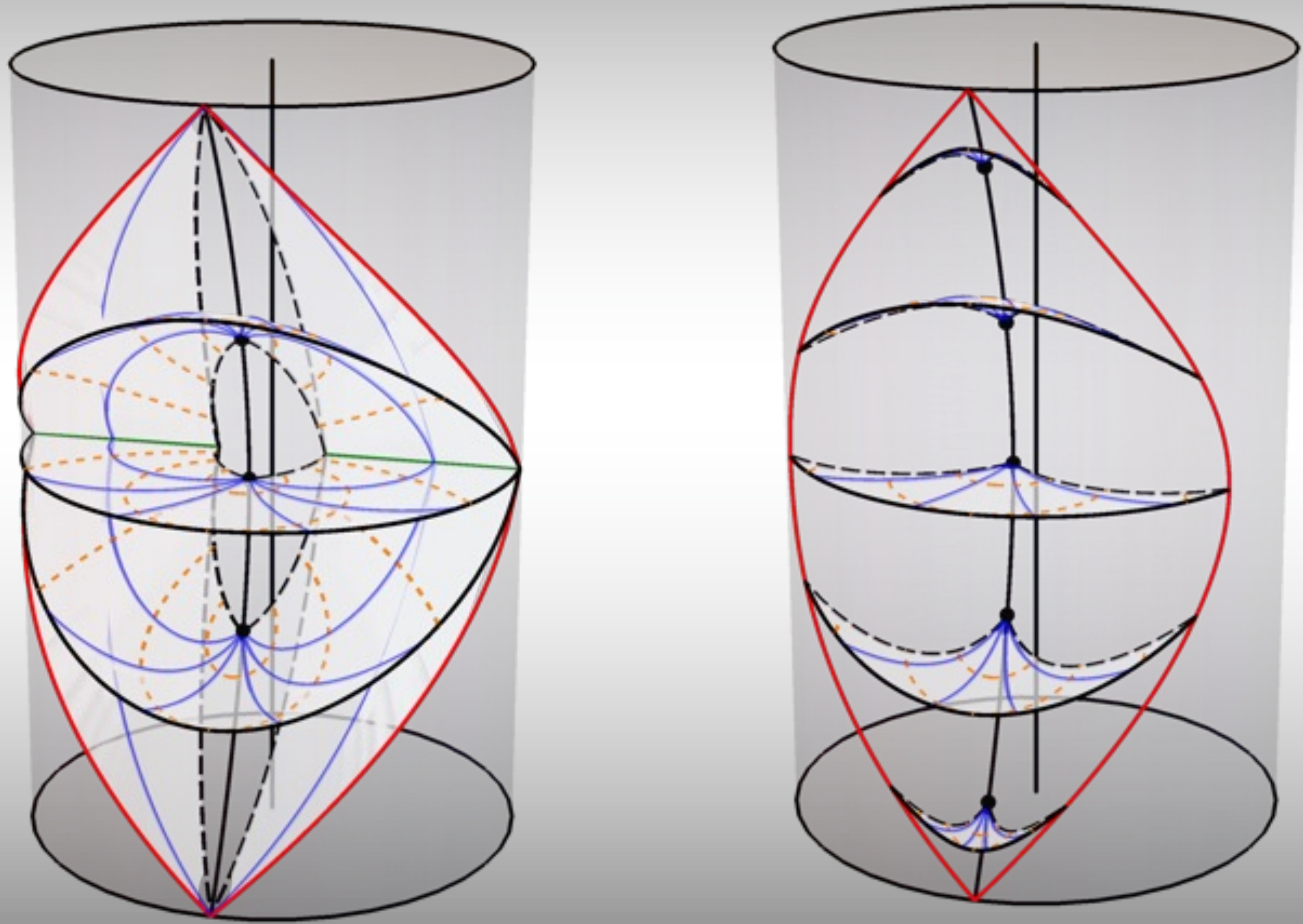


Main difference to 4D: no accelerating partner!

Rapidly accelerating heavy Particle



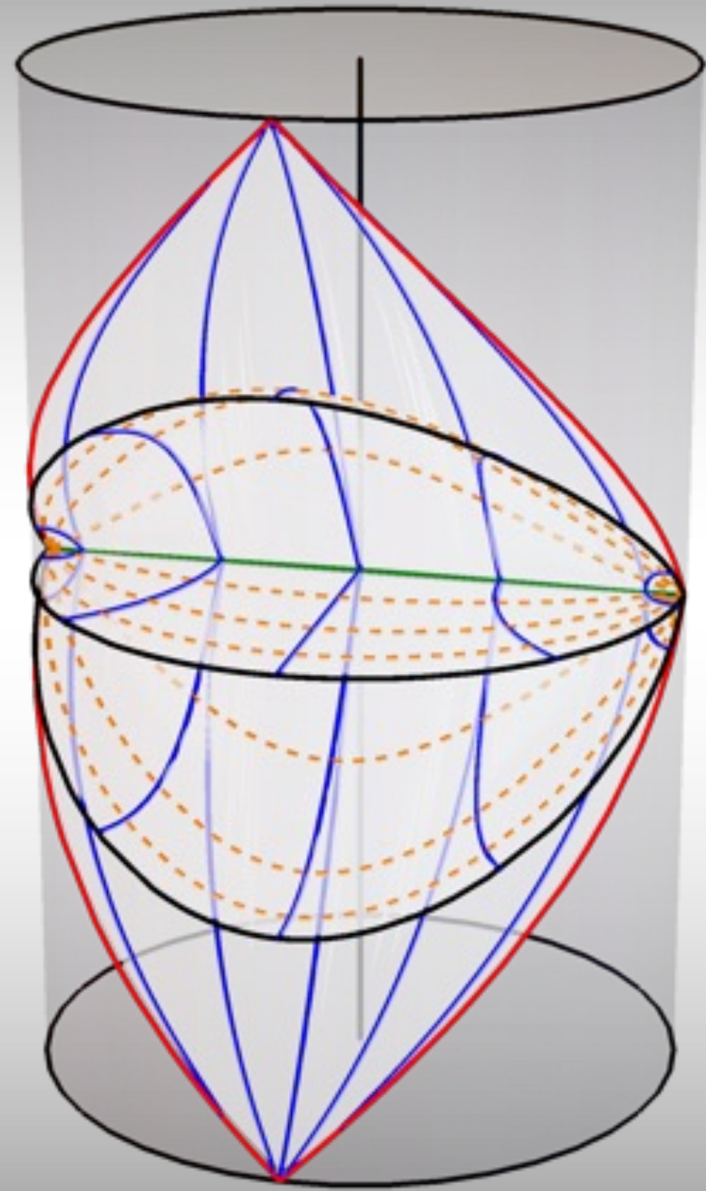
Acceleration with Struts



BTZ

Recall the BTZ black hole is an identification of the Rindler wedge:

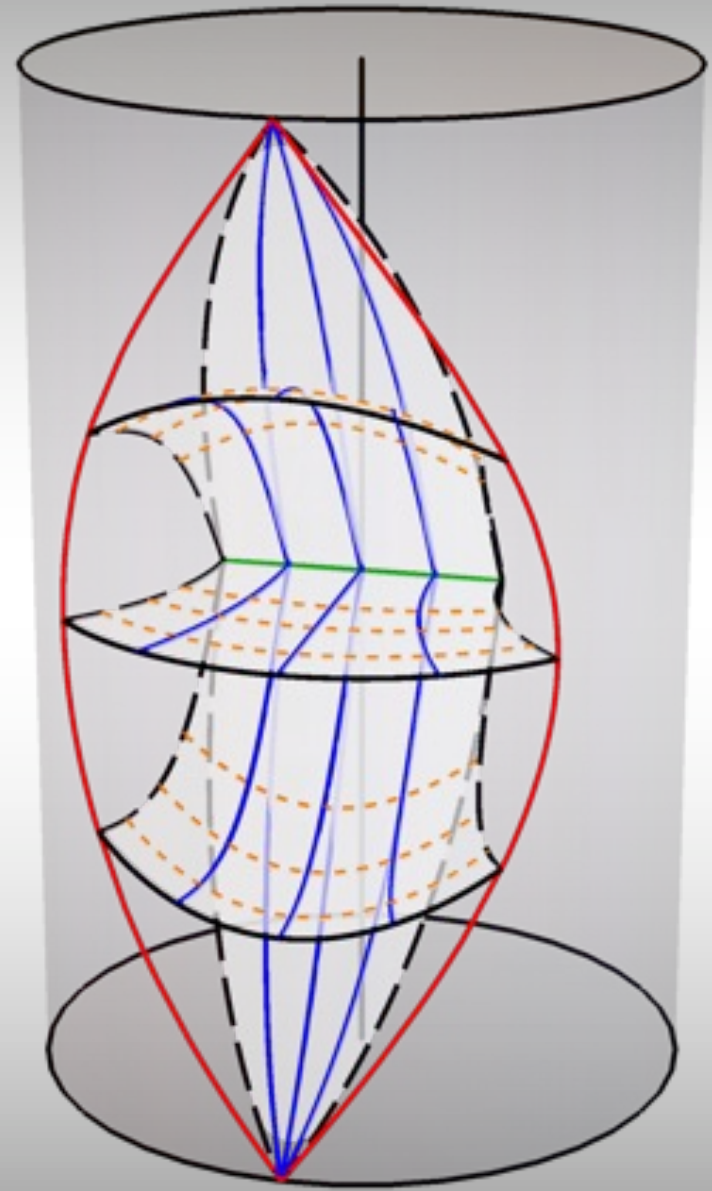
Blue lines are constant ϕ, and have zero extrinsic curvature,



BTZ

Recall the BTZ black hole is an identification of the Rindler wedge:

Blue lines are constant ϕ, and have zero extrinsic curvature, so can cut and paste along ϕ–lines to form the BTZ black hole.



BTZ's as class II

Looking at BTZ from the exact solution perspective:

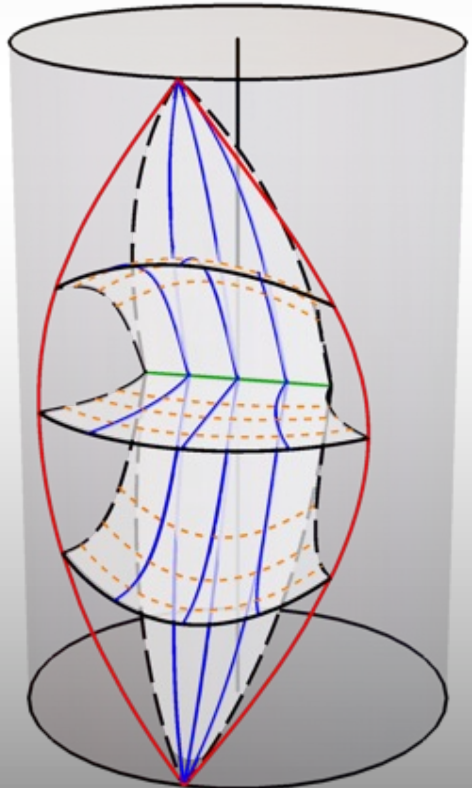
$$ds^2 = \frac{1}{\Omega(r,\psi)^2}\left[F(r)\frac{d\tilde{t}^2}{\alpha^2} - \frac{dr^2}{F(r)} - r^2 d\psi^2\right],$$

$$F(r) = -m^2(1-\mathcal{A}^2 r^2) + \frac{r^2}{\ell^2},$$

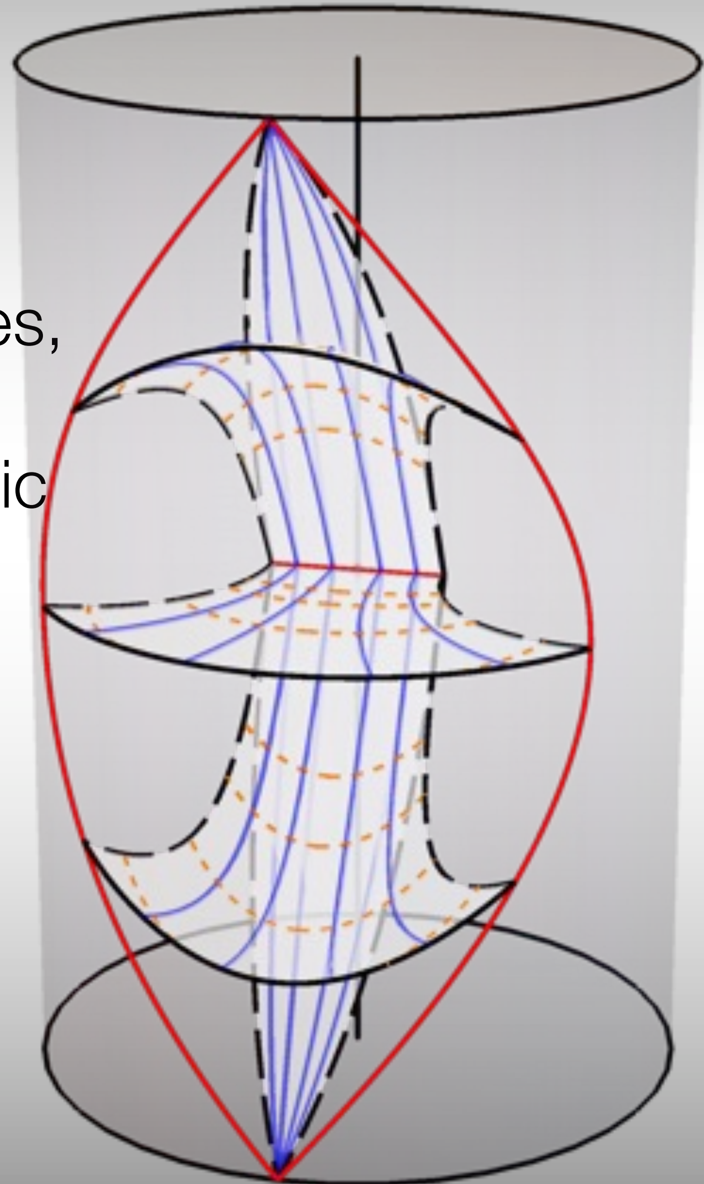
$$\Omega(r,\psi) = 1 + \mathcal{A} r \cosh(m\psi)$$

$$\begin{pmatrix} K = 1/m \\ A = m\mathcal{A} \end{pmatrix}$$

BTZ's

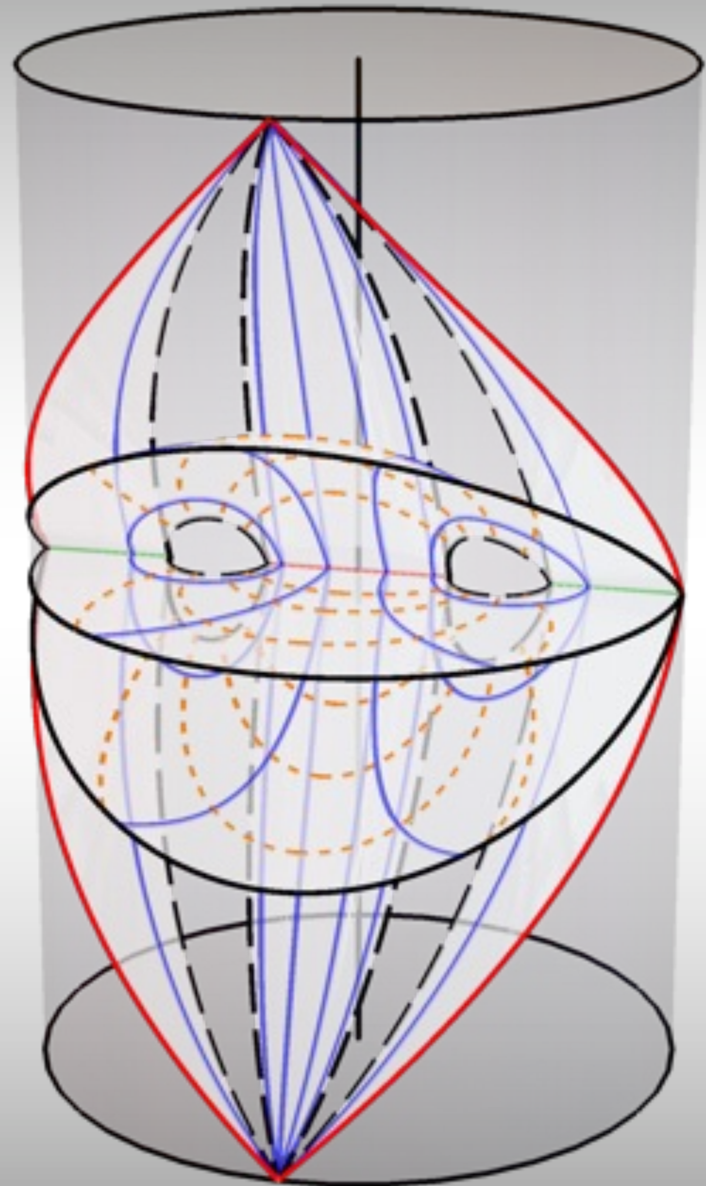


Adding A in the class 2's skews the constant ϕ lines, changing the way AdS is sliced and adding extrinsic curvature to constant ϕ–lines – here is a slightly distorted BTZ (slow acceleration)



Rapid BTZ's

Because the distorted ϕ–lines now wrap back to the Rindler wedge horizon, for some values of ϕ we get an "additional" horizon (different portions of the bulk Rindler horizon).



Novel BTZ

Hiding within class I is a new BTZ-like solution. If Al>1, have a horizon at

$$y_h^2 = 1 - \frac{1}{A^2\ell^2}$$

For the accelerating particle, we usually take y<-y_h with y ~ -1/Ar, but can also have $y \in (y_h, x) \qquad x \in (x_+, 1)$

To make this look more familiar, take

$$\tau = \frac{At}{\alpha}, \qquad y = \frac{1}{A\rho}, \qquad x = \cos(\phi/K)$$

where

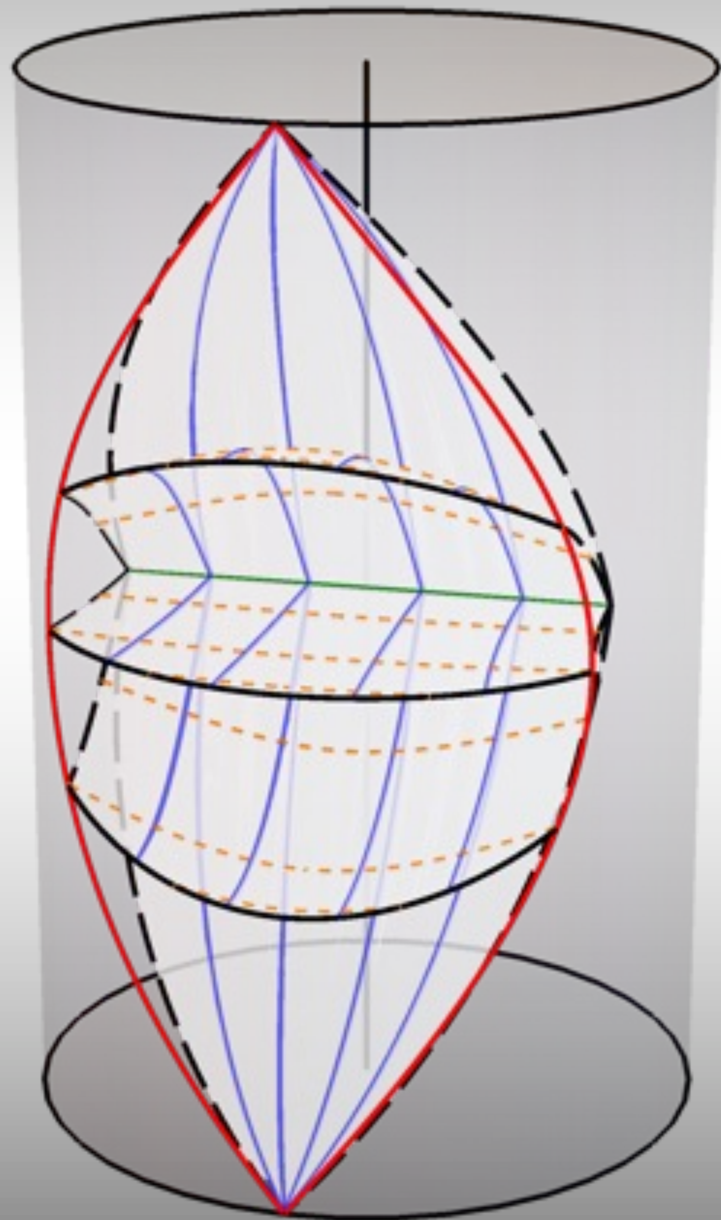
$$K = \pi / \arccos(x_+) > \pi / \arccos(y_h) > 2$$

So that this solution is clearly disconnected from the non-accelerating solutions.

$$ds^2 = \frac{1}{\left[A\rho\cos\left(\frac{\phi}{K}\right) - 1\right]^2}\left(f(\rho)\frac{dt^2}{\alpha^2} - \frac{d\rho^2}{f(\rho)} - \rho^2\frac{d\phi^2}{K^2}\right)$$

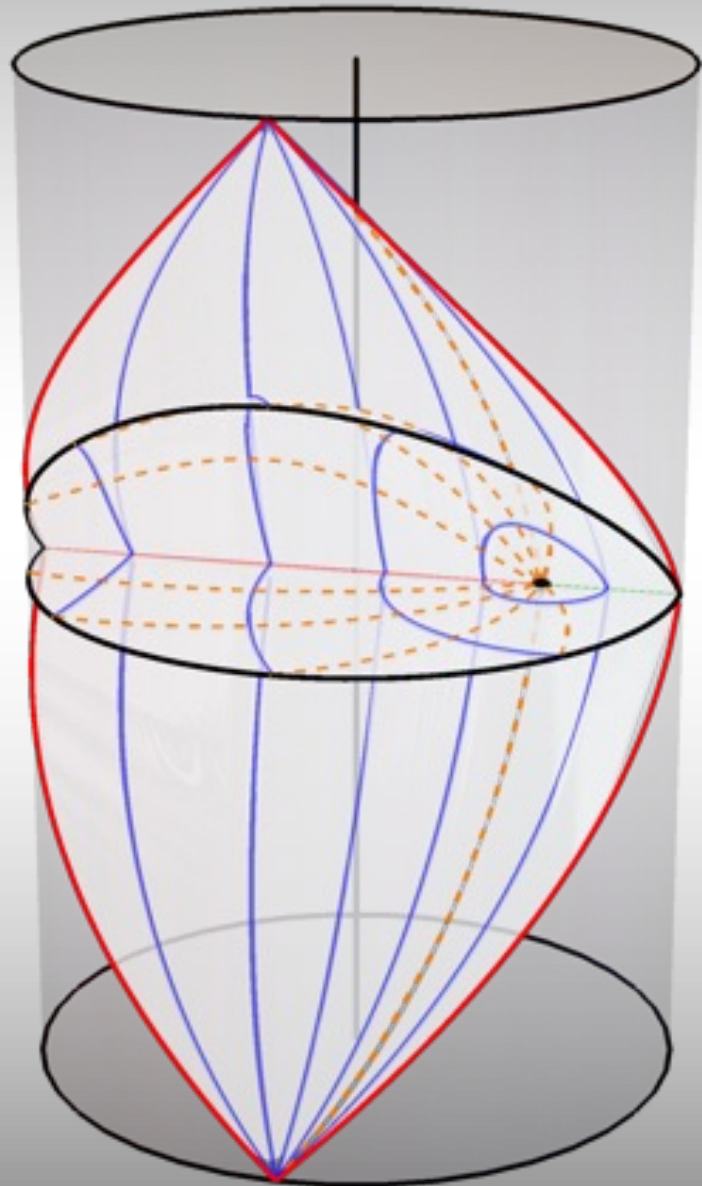
$$f(\rho) = 1 - (A^2\ell^2 - 1)\rho^2/\ell^2$$

Plotting this solution in global coordinates shows a clear parallel with BTZ. This time however, there is no continuous link to the BTZ metric.

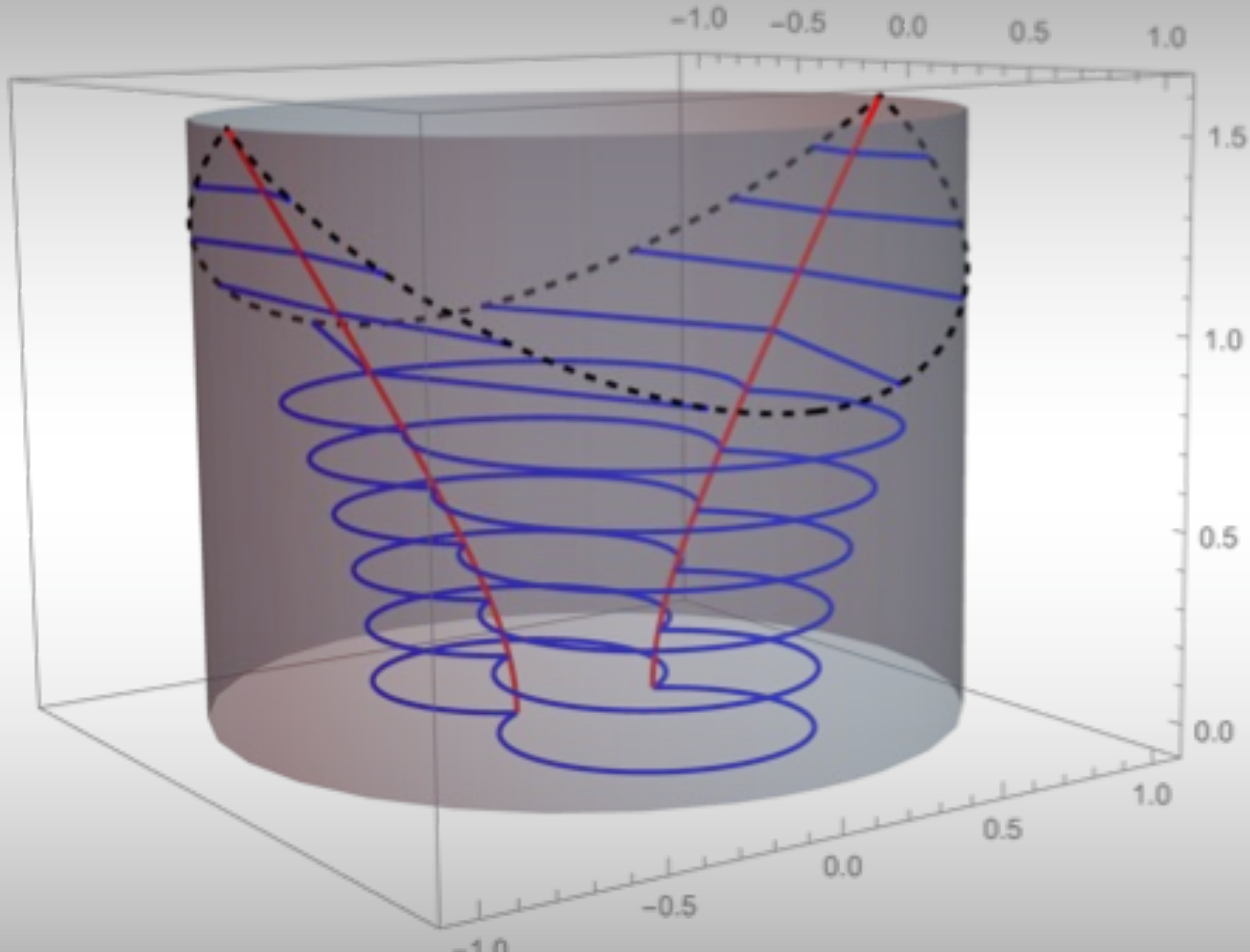


Class III - braneworld

Finally, the class 3 solutions don't allow for an identification of a single bulk with a wall, instead we take 2 bulk copies a la Randall-Sundrum to form a braneworld.



Rapidly accelerating heavy particle – full bulk.



Would like to understand bulk and holographic nature of 3D solutions, as well as thermal back-reaction.

Recap

- Have shown how to allow for varying tension in thermodynamics of black holes.
- Conjugate variable is *Thermodynamic Length*
- Thermodynamics of accelerating black holes is computable – non-static and non-isolated.
- A key technical point is the normalisation of timelike Killing vector
- Have derived extensive expressions for the TD variables and a new Reverse Isoperimetric Inequality.
- Three dimensions is both familiar and new!