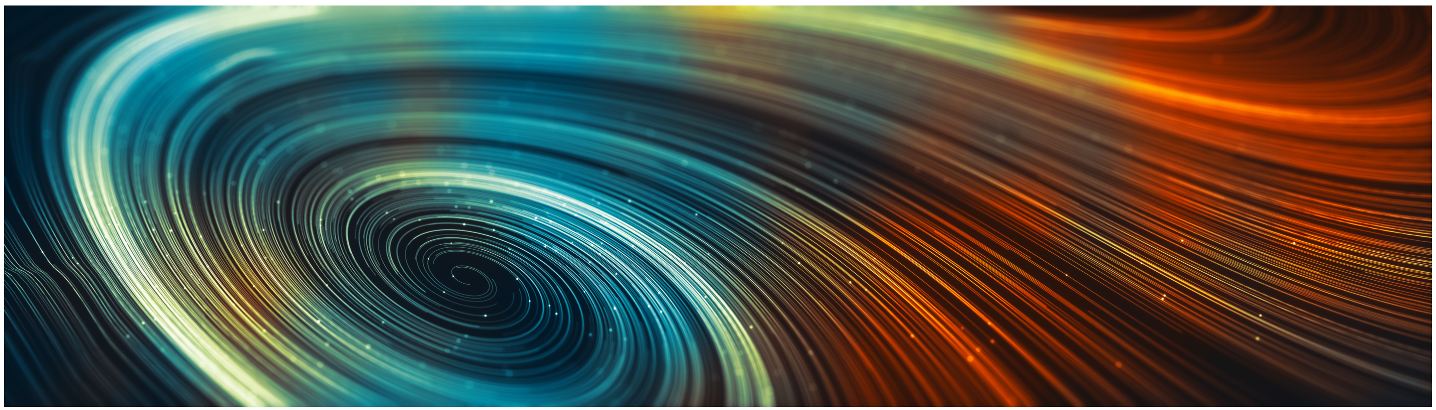
# Relativistic fluids with a twist



Nils Andersson

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University of Southampton

# Erice 1991

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Erice Lecture Notes

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B. Carter
May 1991

# MECHANICS AND EQUILIBRIUM GEOMETRY OF HORIZONS, MEMBRANES AND STRINGS

Part I MECHANICS AND EQUILIBRIUM OF BLACK HOLE HORIZONS
(Ref "Gravitation in Astrophysics" ed. B.Carter, J.B.Hartle, p63, Plen...

Part II COVARIANT MECHANICS OF ME...
(Ref "Formation and Evolution of C... Cambridge 1989, ed. G.W. Gibbons, S... Vachaspati, Cambridge Un...

## Introductory Remark

To understand or control ... one grasps for what is (at ... conserved, whence importanc... stable equilibrium states o... and ... beyond?

Chandrasekhar's (... mass limit for ordinary ... equilibrium of cold (wh... raised problem posed in ... Cosmic Censorship postul... states – for which stat... of Outer Communicati... horizon hiding singul...

Part I pr... General Relativist... Equilibrium Stat... on uniqueness ... of Kerr vaccum ... stability and ... more specialis...

Part ... mechanics incl... (timelike anal... to their equ...

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p1

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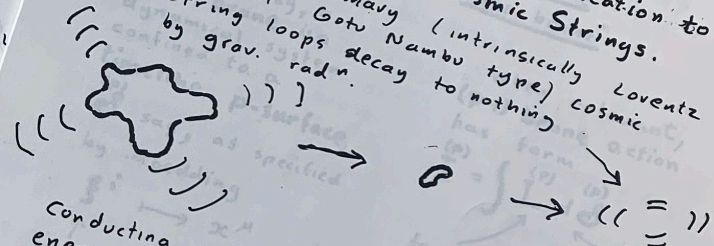
May, 1991.

Part II

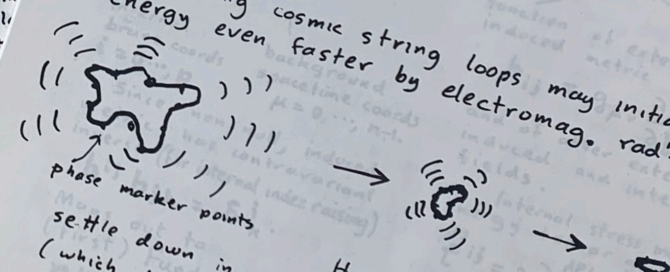
# COVARIANT MECHANICS OF MEMBRANES, STRINGS, ...

One particular motivation: application to **Superconducting Cosmic Strings.**

Ordinary (intrinsically Lorentz covariant Goto Nambu type) cosmic string loops decay to nothing by grav. radn.



Conducting cosmic string loops may initially lose energy even faster by electromag. radn. also




However they ultimately settle down in non radiating equilibrium states (which do not exist for non-conducting variety) presumably minimising energy for given values of conserved integrals such as phase winding number. Resulting stationary relic loops (from conducting strings produced at electro weak unification) may be of cosmological importance.

# EU Network meeting 2002


William Hodge, Pembroke College

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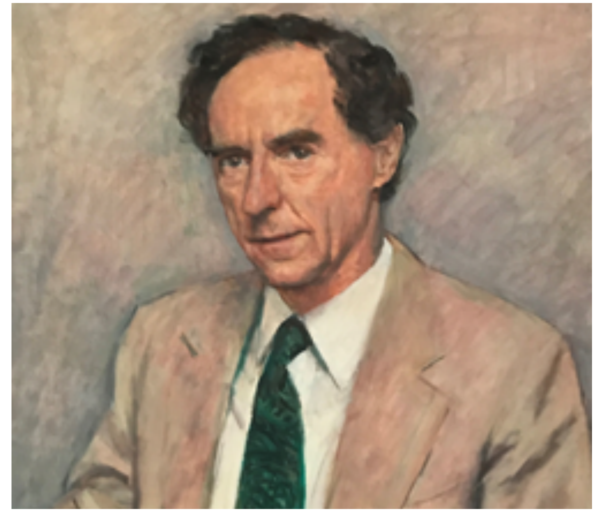
George Batchelor, DAMTP

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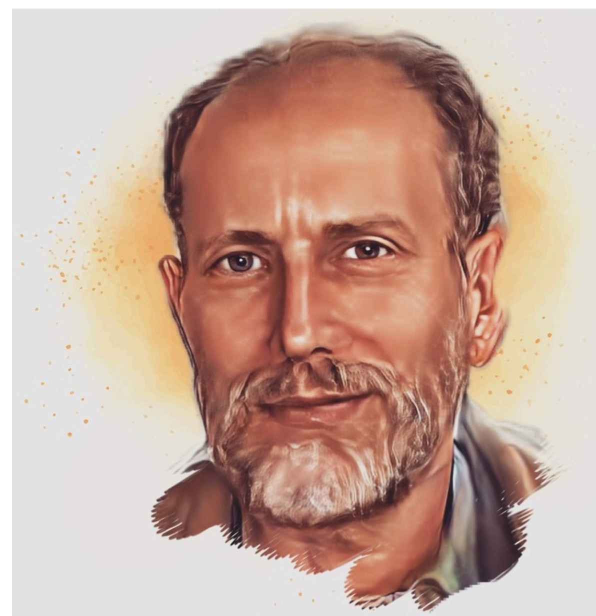

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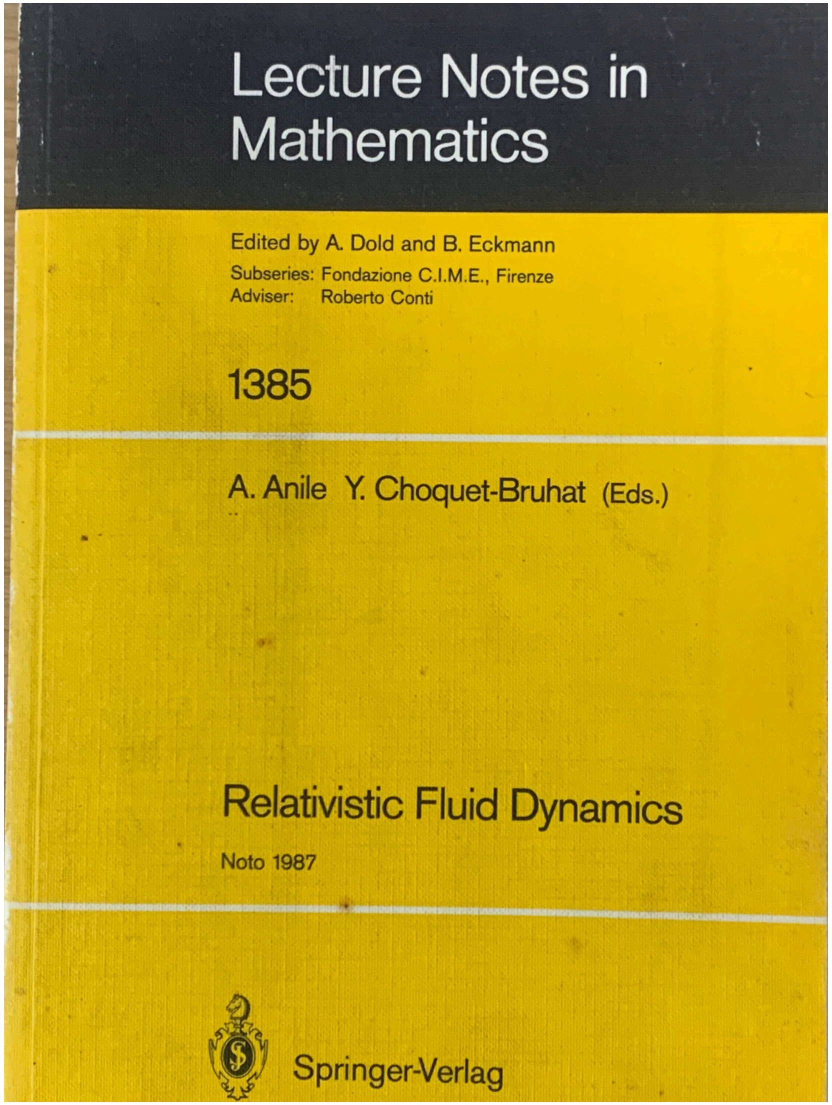


# COVARIANT THEORY OF CONDUCTIVITY IN IDEAL FLUID OR SOLID MEDIA.

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REVIEW ARTICLE

# Relativistic fluid dynamics: physics for many different scales

Nils Andersson[1] · Gregory L. Comer[2]

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Archimedes
(287-212 BC)

**EARLY EMPIRICAL STUDIES**

Leonardo da Vinci
(1425-1519)

**1700s FLUID DYNAMICS**

Isaac Newton
(1642-1727)

Daniel Bernoulli
(1700-1782)

**1800s VISCOSITY**

Claude Louis Navier
(1785-1836)

George Gabriel Stokes
(1819-1903)

**1900s RELATIVITY**

Albert Einstein
(1879-1955)

**1950s SUPERFLUIDS**

Heike Kamerlingh-Onnes
(1853-1926)

Lev Davidovich Landau
(1908-1968)

**MULTIFLUID MODELS**

Isaak Markovich Khalatnikov
(1919-)

Brandon Carter
(1942-)

**VARIATIONAL METHODS**

Leonhard Euler
(1707-83)

Louis de Lagrange
(1736-1813)

Sophus Lie
(1842-1899)

**VORTICITY**

Hans Ertel
(1904-1995)

Andre Lichnerowicz
(1915-1998)

**IRREVERSIBLE THERMODYNAMICS**

Lars Onsager
(1903-1976)

Ilya Prigogine
(1917-2003)

**RELATIVISTIC DISSIPATION**

Carl Henry Eckart
(1902-1973)

Werner Israel
(1931-)

John M. Stewart
(1943-2016)

# convective variations

**Convective variational principle** makes use of the idea of a three dimensional "matter space".

For a single fluid Lagrangian $\Lambda=\Lambda(n^2)$ where $n^2=-n^a n_a$ we have the conjugate momentum

$$\mu_a = -2\frac{\partial\Lambda}{\partial n^2} n_a = B n_a$$

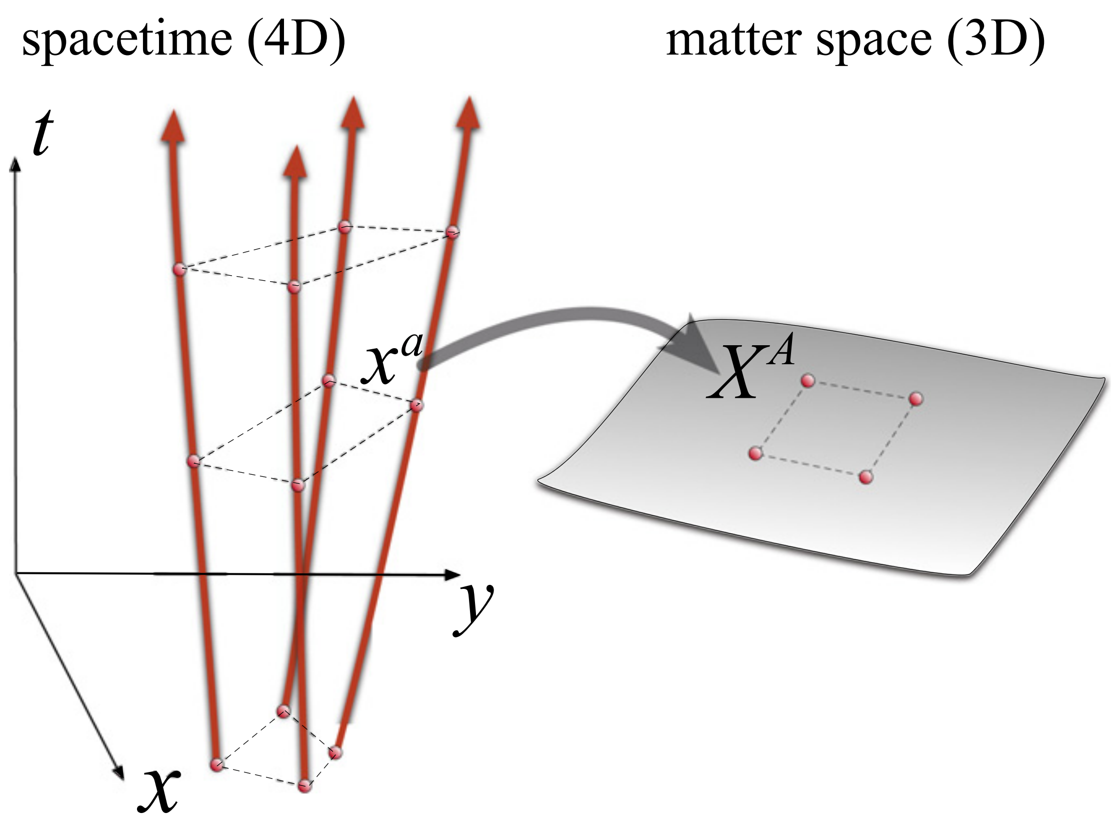
A general variation leads to

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left[\mu_a \delta n^a + \frac{1}{2}\left(\Lambda g^{ab} + n^a \mu^b\right)\delta g_{ab}\right]$$

which means that the momentum has to vanish! Clearly not what we want.

Need a **constrained variation**, ensuring a conserved particle flux. This follows if the associated volume form $n_{abc} = \varepsilon_{abcd} n^d$ is closed, i.e.

$$\nabla_{[a} n_{bcd]} = 0 \quad \Rightarrow \quad \nabla_a n^a = 0$$




Introduce matter space with coordinates $X^A$ and associated displacement such that

$$\Delta X^A = 0 \quad \Rightarrow \quad \delta X^A = -\xi^a \nabla_a X^A$$

Then the variation leads to

$$\delta\left(\sqrt{-g}\Lambda\right) = \sqrt{-g}\left[\frac{1}{2}\left(\Psi g^{ab} + n^a \mu^b\right)\delta g_{ab} - f_a \xi^a\right]$$

where $\Psi = \Lambda - n^a \mu_a$.

# many fluids

The final equations of motion take the form:

$$f_a = 2n^b \nabla_{[b}\mu_{a]} = 2n^b \omega_{ab} = 0$$

emphasize the role of the (momentum) vorticity.

Also – and this is crucial – it is now straightforward to account for several identifiable fluxes (=multi-fluid systems!).

"Simply" need several matter spaces (one for each flux);

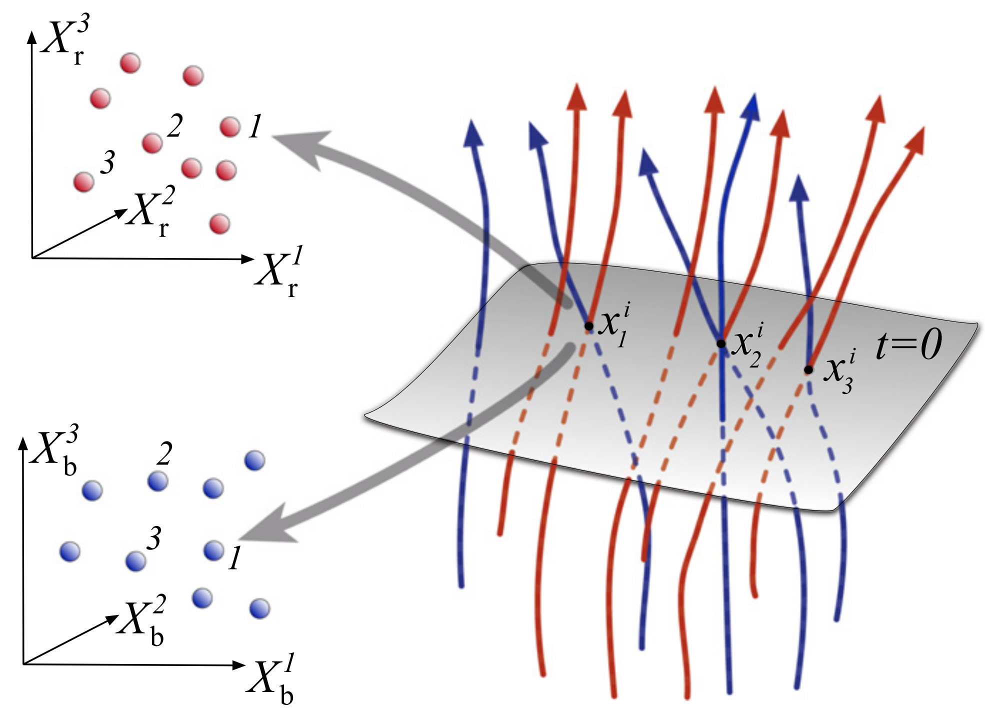
$$n_{\mathrm{x}}^a = n_{\mathrm{x}} u_{\mathrm{x}}^a$$

Following the same procedure as before, we arrive at:

$$\nabla_a n_{\mathrm{x}}^a = 0$$

$$f_a^{\mathrm{x}} = 2n_{\mathrm{x}}^b \nabla_{[b}\mu_{a]}^{\mathrm{x}} = 2n_{\mathrm{x}}^b \omega_{ba}^{\mathrm{x}} = 0$$

(no summation over x).



# why bother?

- "mixtures", where the components retain their identity (chemistry).
- superfluids, where finite temperature excitations (e.g. phonons) lead to a two-fluid model (condensed matter physics)
- heat flow, where the entropy (say, phonons) may be treated as a "fluid" (non-equilibrium thermodynamics)
- electromagnetism, where a charge current is required to maintain the magnetic field (plasma physics)
- elasticity, where the matter space approach provides a natural description of deformations (solid state physics)

Strong motivation for trying to understand these systems in general relativity, ranging from neutron star astrophysics to early Universe cosmology.

Interesting connections with laboratory atomic Bose-Einstein condensates, where the Hamiltonian can be "designed to specification".

Depending on the situation under consideration, the force $f_a^{\mathrm{x}}$ encodes additional physics, like dissipation, elasticity, mutual interaction forces…

# towards dissipation

In order to model "reality", we need to account for dissipation (friction, thermal conductivity, resistivity etc).

Disregarding conventional wisdom, according to which action principles do not exist for dissipative systems, we have extended the variational approach in this direction.

How? The key conceptual step involves breaking the closure of the individual volume forms.

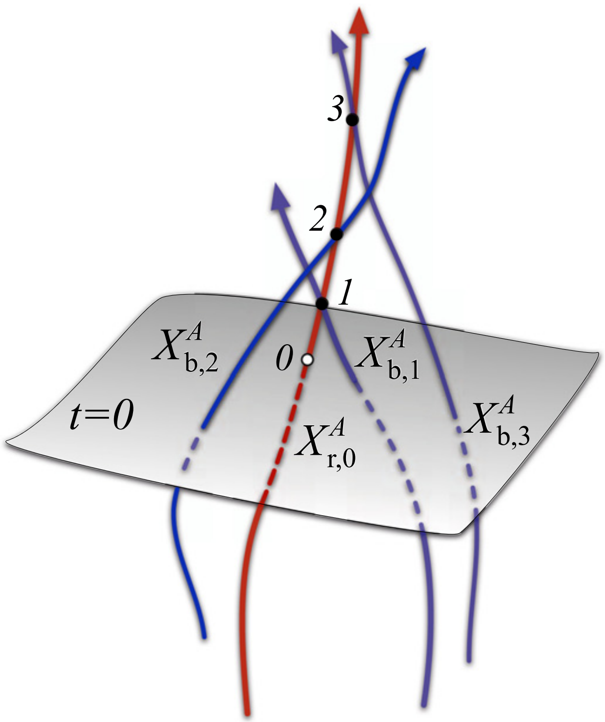
For example, if each $n^{\mathrm{x}}_{abc}$ is no longer just a function of its own $X^A_{\mathrm{x}}$, the closure will be broken. As the fluxes are no longer conserved, the formalism incorporates dissipation (in some sense). In general, we have

$$\delta n^{\mathrm{x}}_{abc} = -\mathcal{L}_{\xi_{\mathrm{x}}} n^{\mathrm{x}}_{abc} + \frac{\partial X^A_{\mathrm{x}}}{\partial x^{[a}} \frac{\partial X^B_{\mathrm{x}}}{\partial x^{b}} \frac{\partial X^C_{\mathrm{x}}}{\partial x^{c]}} \Delta_{\mathrm{x}} n^{\mathrm{x}}_{ABC}.$$

In practice, we may let each volume form depend on:

(1) the coordinates of all the matter spaces, and

(2) the independent mappings of the spacetime metric into these spaces.




**Intuition:** The individual matter space coordinates do not vary along their own world lines, even when the system is dissipative. By adding a dependence on the other matter spaces there is "evolution" since the world lines cut across each other.

**Application:** When **each** volume form depends on **all** sets of matter space coordinates we arrive at a model for reactive/resistive systems.

$$R_a^{xy} \equiv \frac{1}{3!}\mu_x^{ABC}\frac{\partial n^x_{ABC}}{\partial X_y^D}\partial_a X_y^D \qquad R_a^x = \sum_{y\neq x}\left(R_a^{yx} - R_a^{xy}\right)$$

and, introducing the individual creation/destruction rates

$$\Gamma_x = \nabla_a n_x^a$$

it follows that

$$\boxed{2n_x^a\,\nabla_{[a}\mu^x_{b]} + \Gamma_x \perp^a_{xb}\,\mu_a^x = \perp^a_{xb}\,R_a^x}$$

## take home message

# Happy Birthday!