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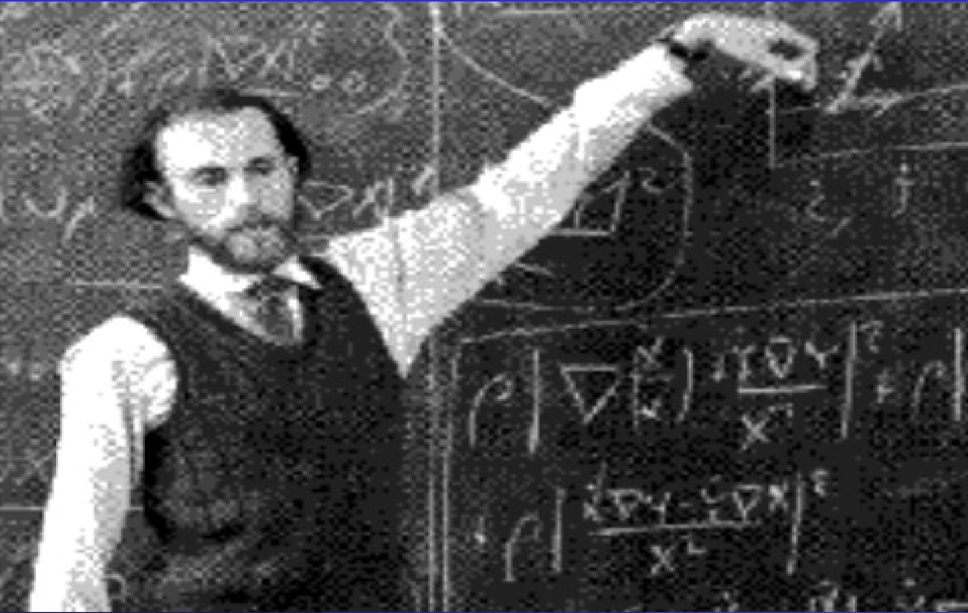

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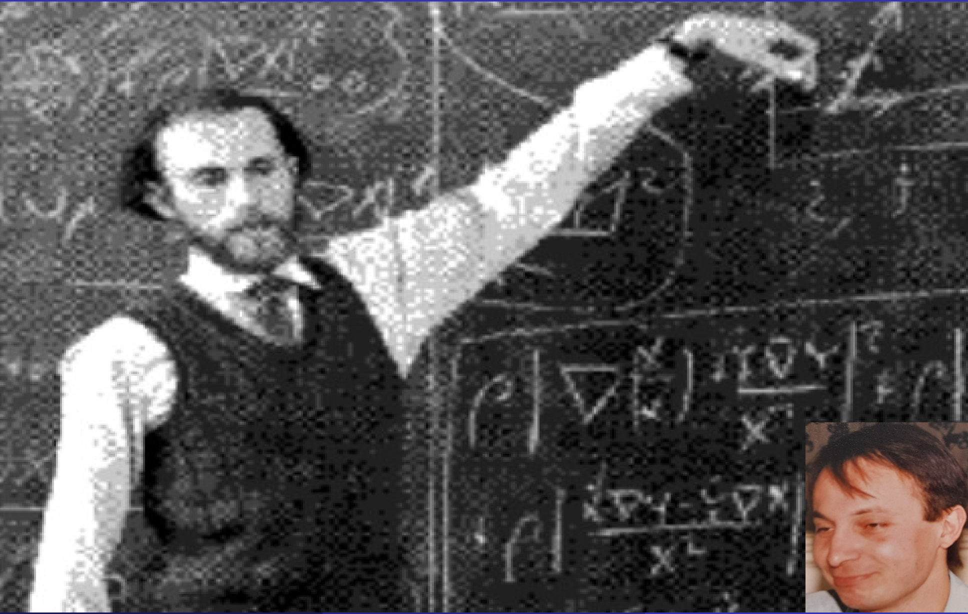

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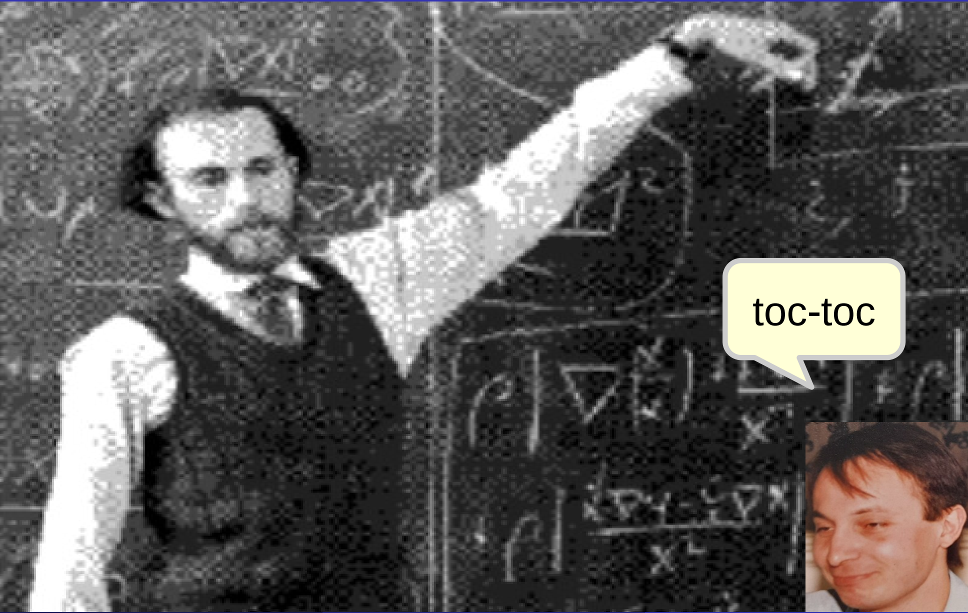

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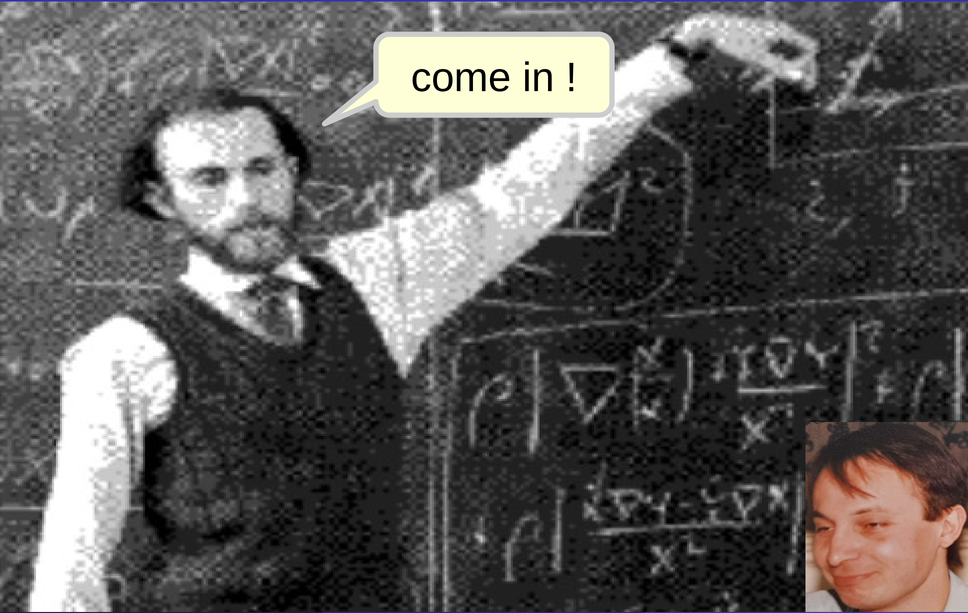

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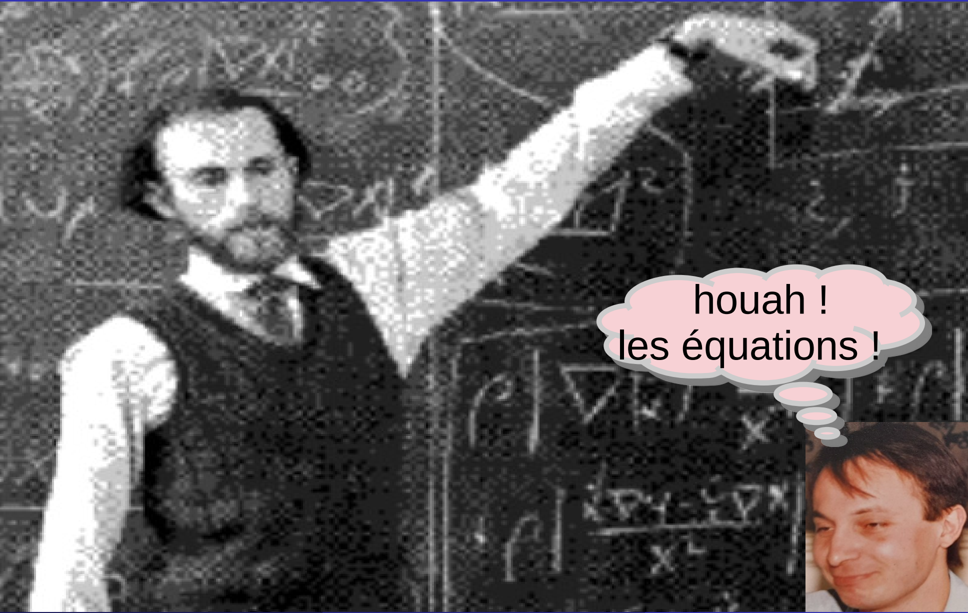

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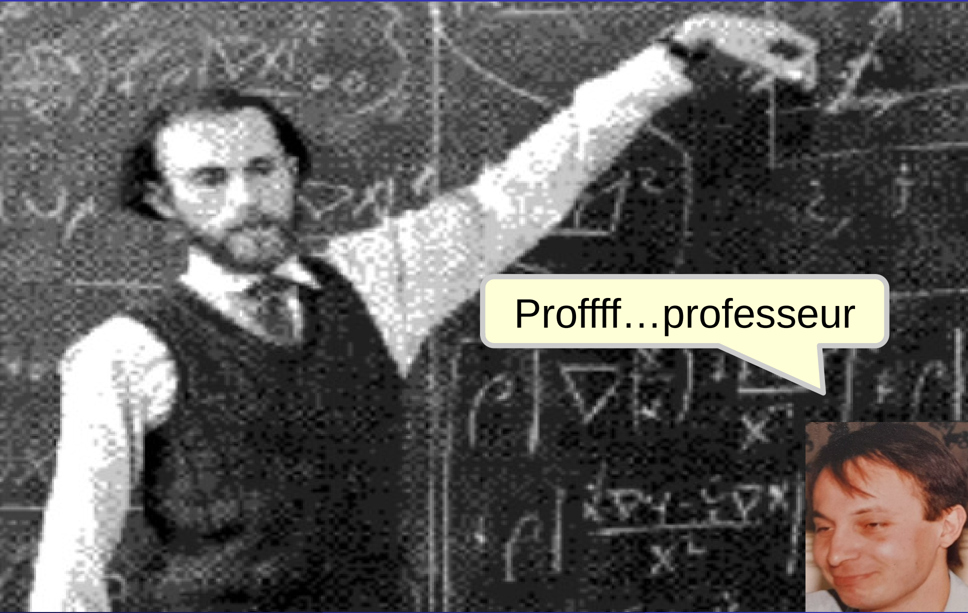

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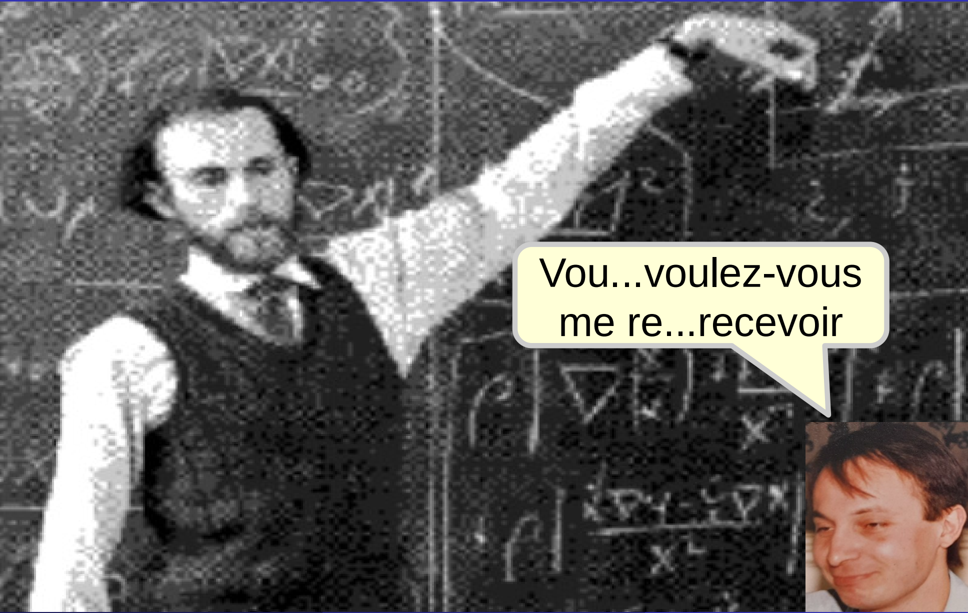

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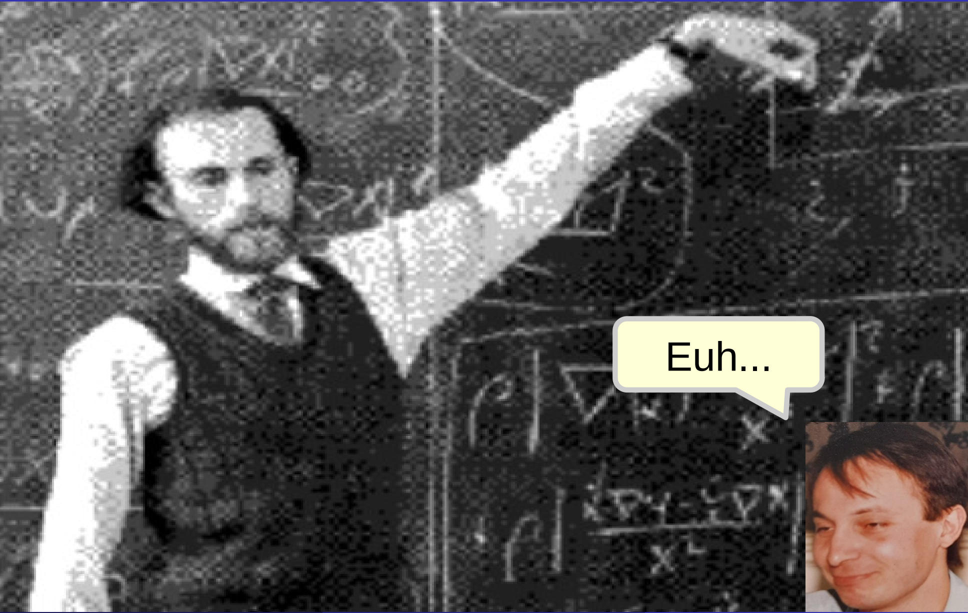

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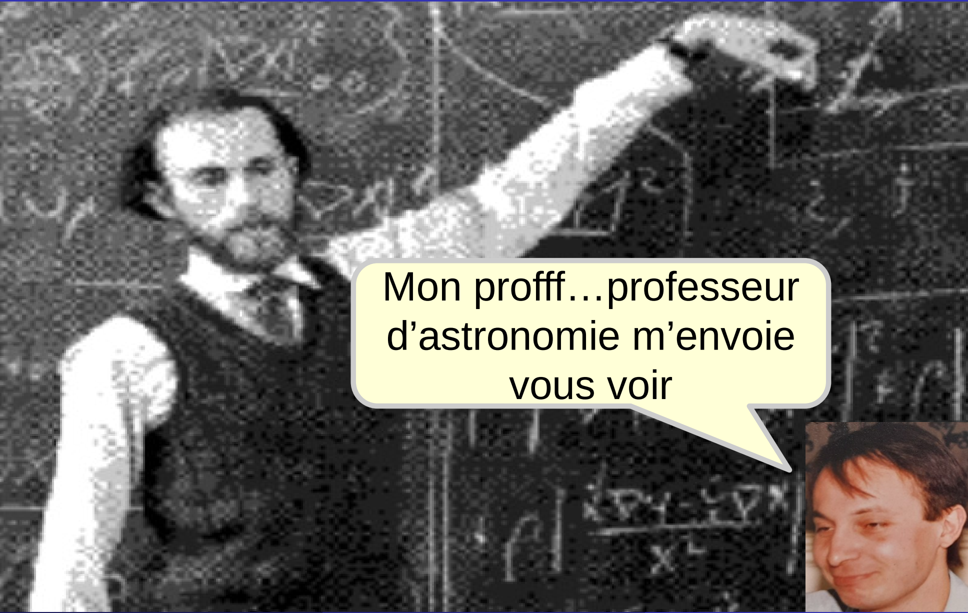

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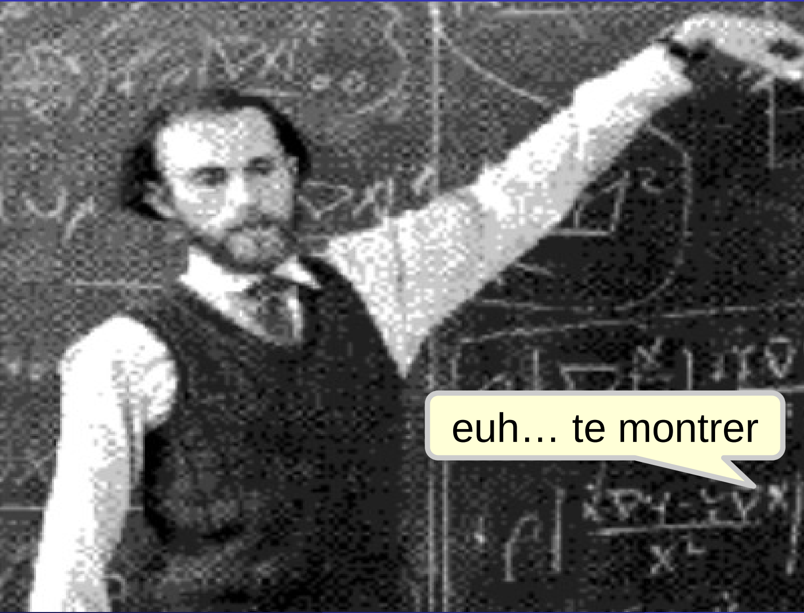

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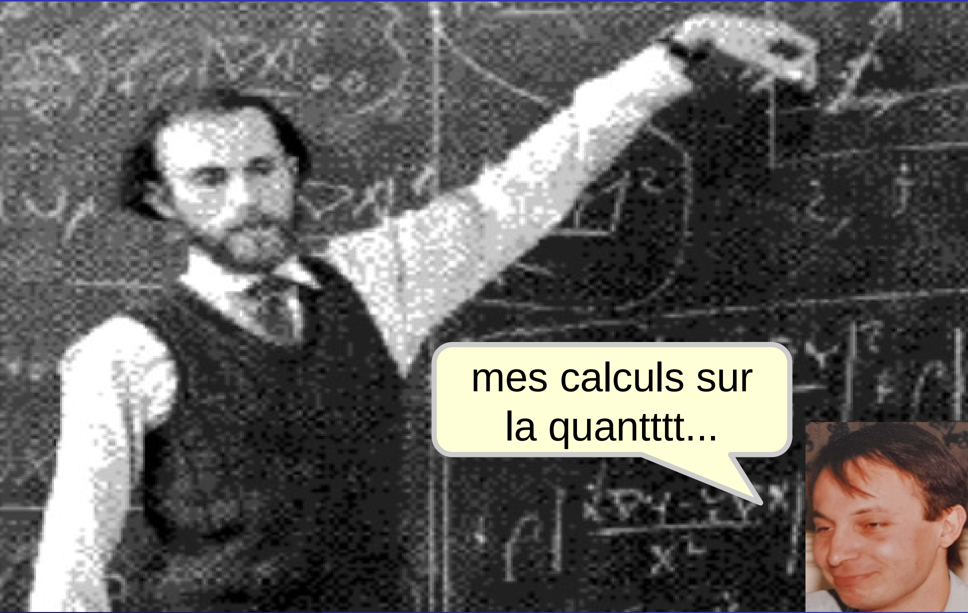

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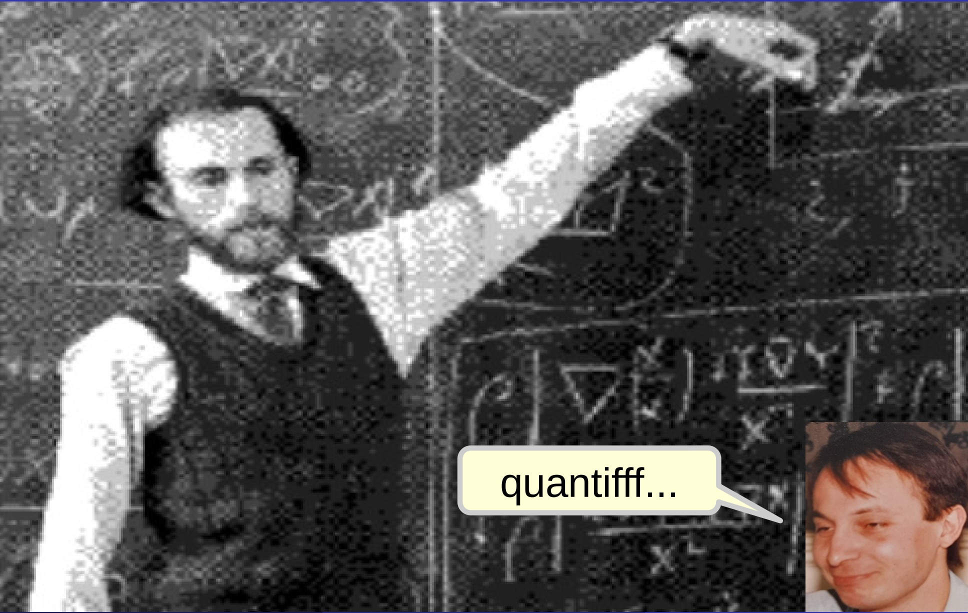

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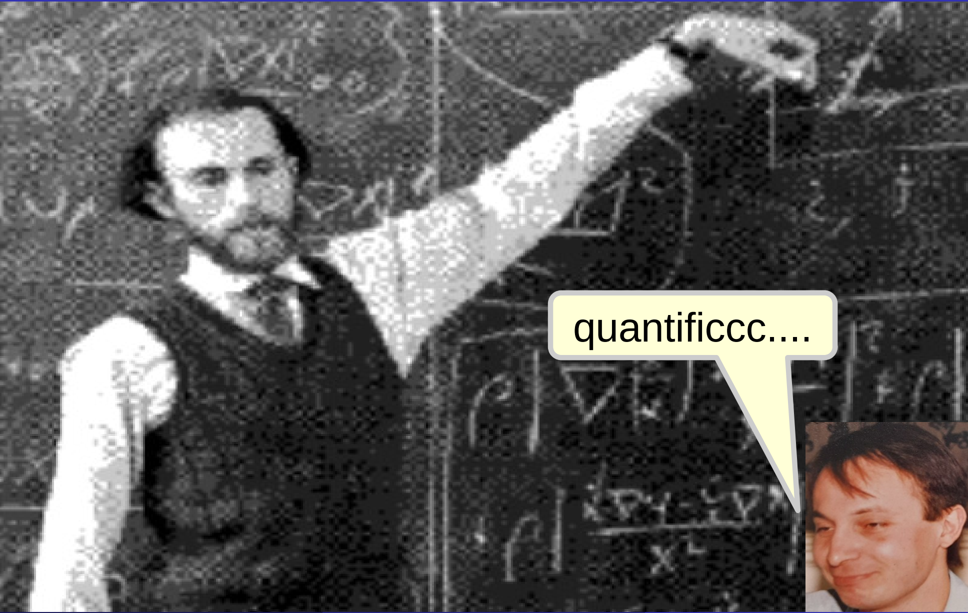

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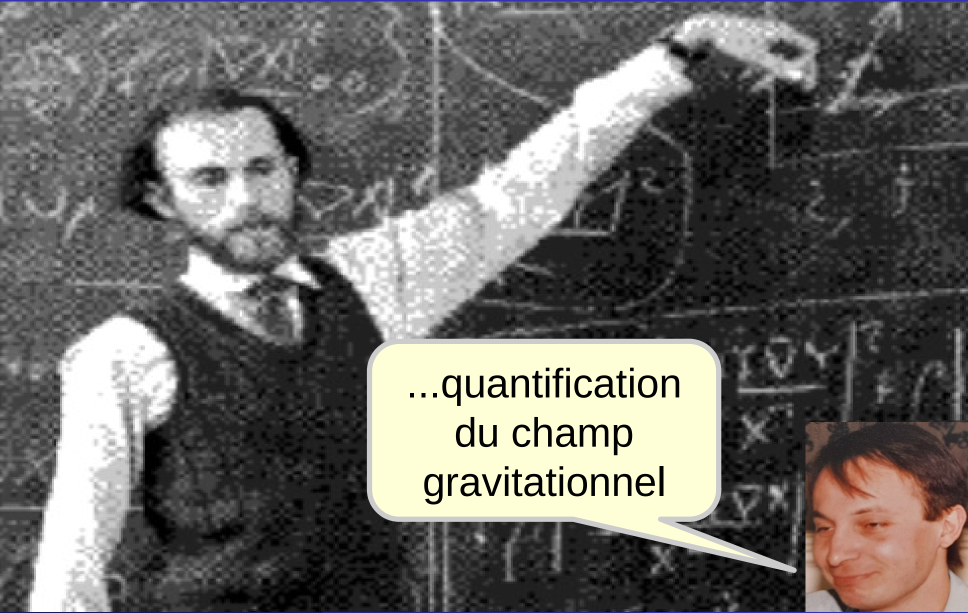

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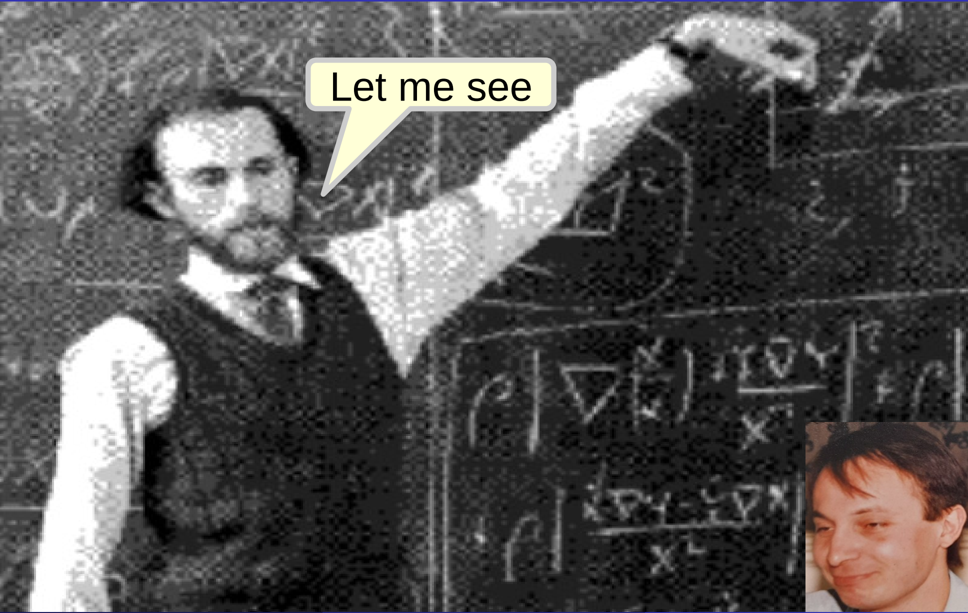

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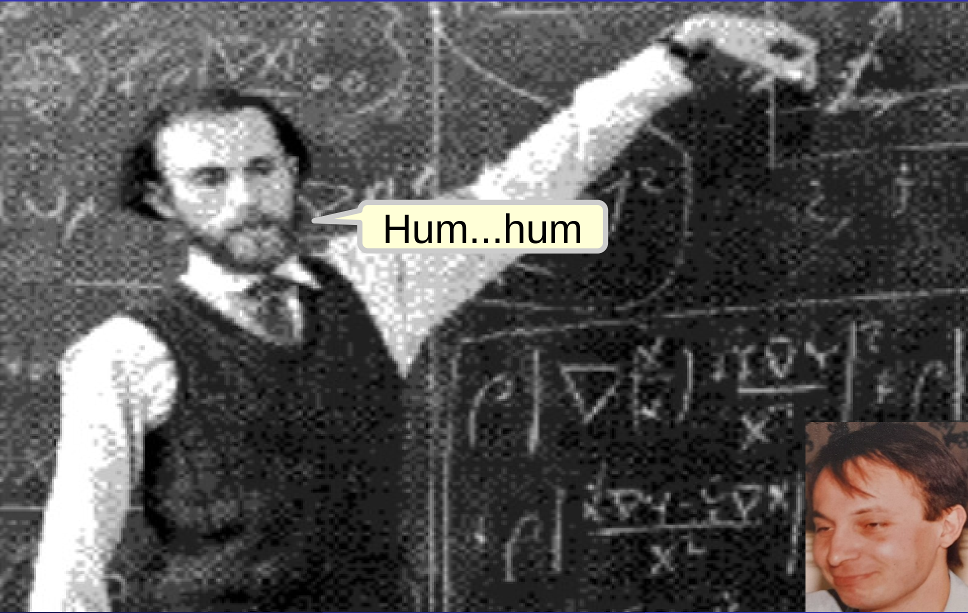

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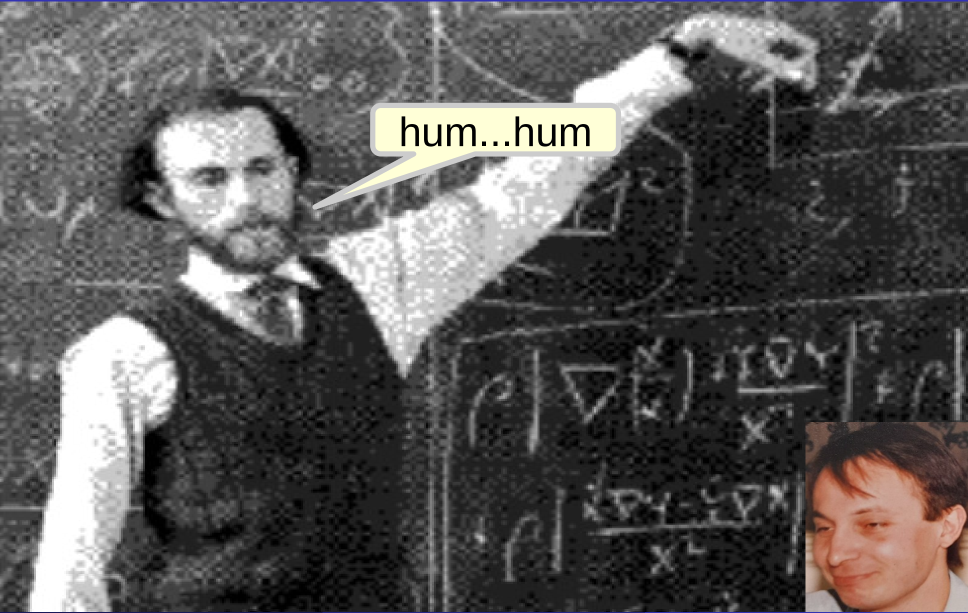

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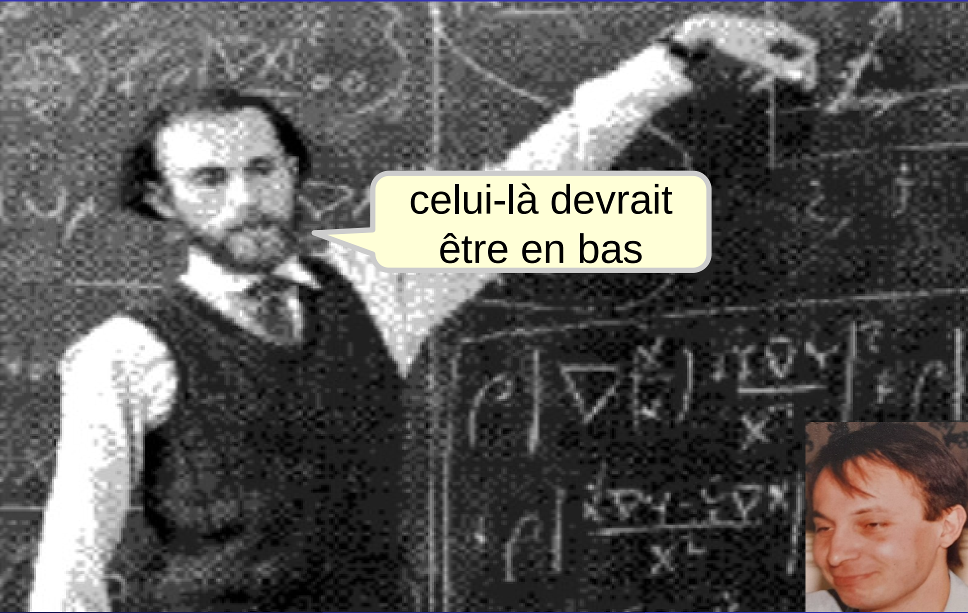

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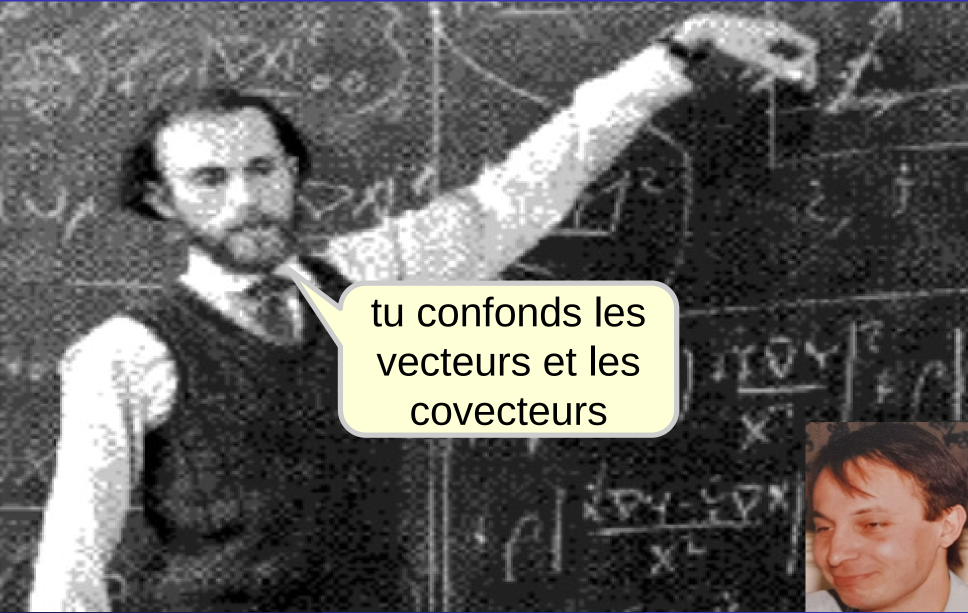

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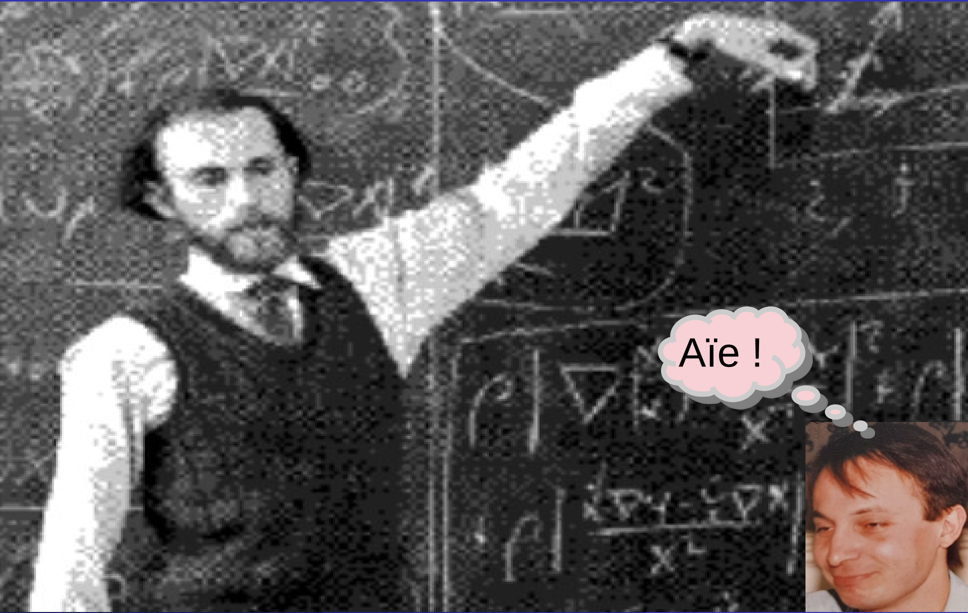

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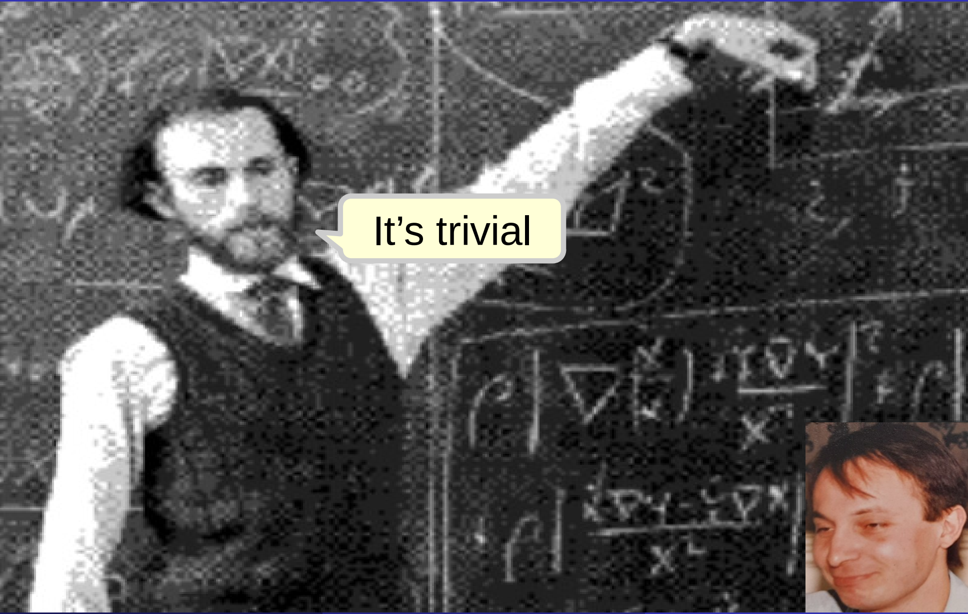

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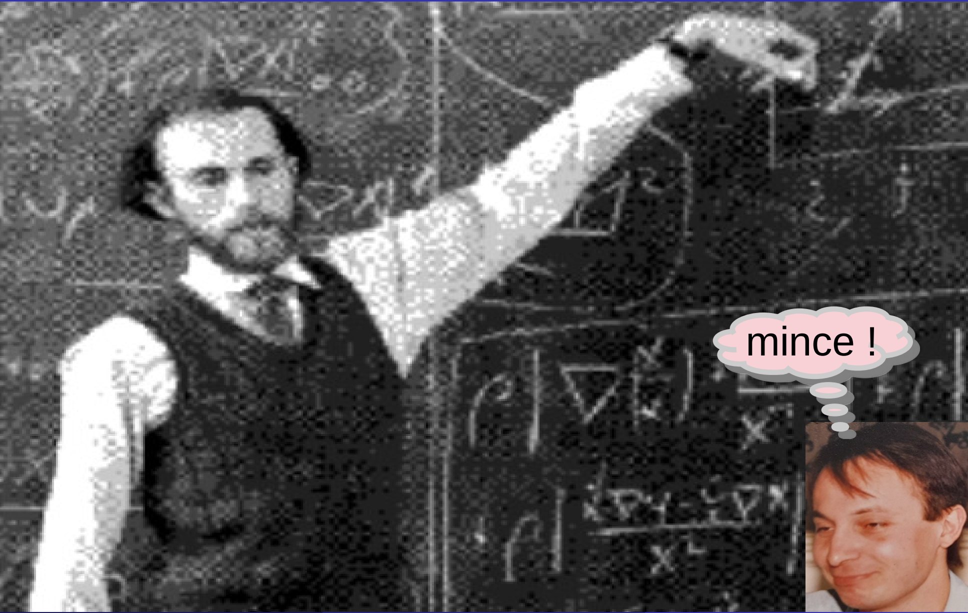

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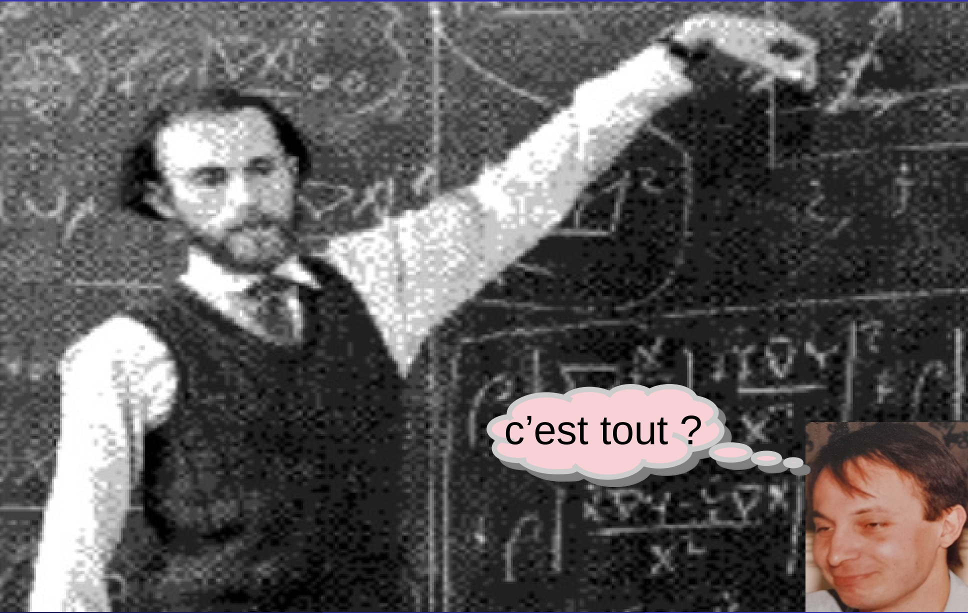

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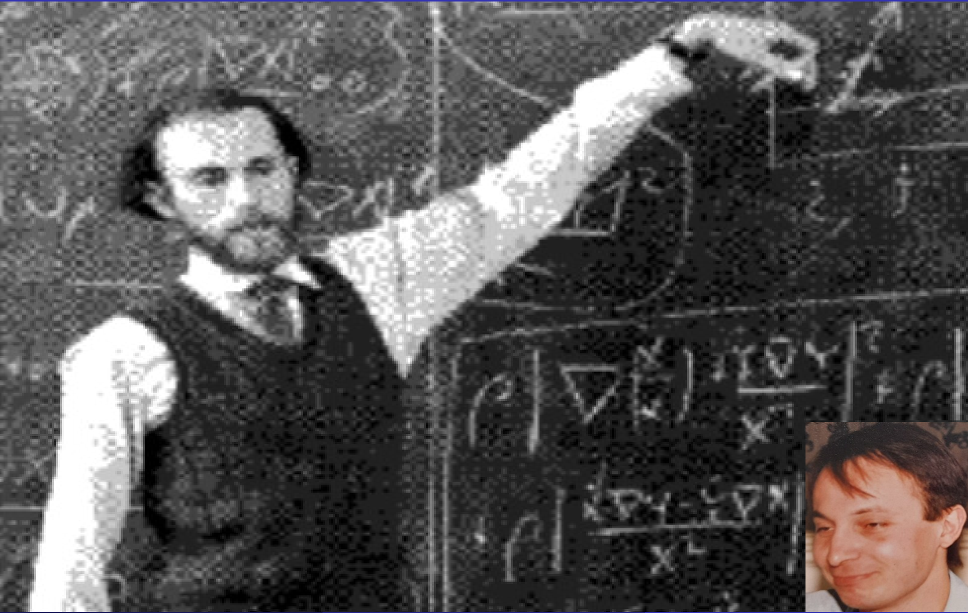

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Carter Fest: Black Holes and other Cosmic Systems

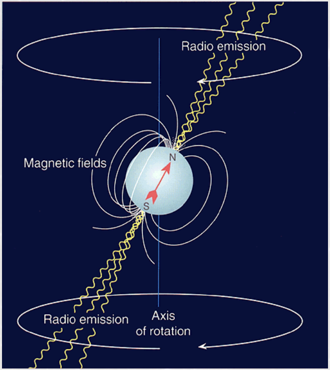
INSPIRALLING BINARY BLACK HOLES

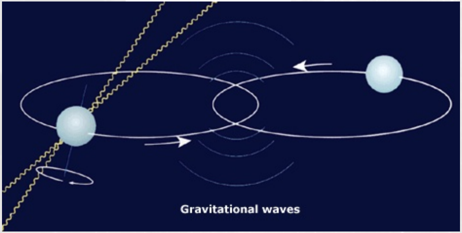
Luc Blanchet

Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris & IPhT

6 juillet 2022

The binary pulsar PSR 1913+16 [Hulse & Taylor 1974]





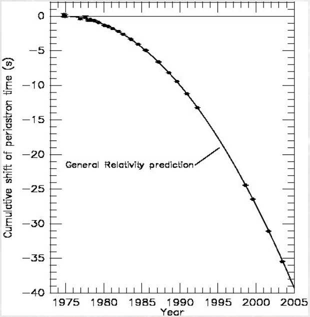
- The pulsar PSR 1913+16 is a rapidly rotating neutron star emitting radio waves like a lighthouse toward the Earth
- This pulsar moves on a (quasi-)Keplerian close orbit around an unseen companion, probably another neutron star

The orbital decay of the Hulse-Taylor binary pulsar

$$4\pi R^2 \mathfrak{J} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu}\dddot{\mathfrak{J}}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu}\dddot{\mathfrak{J}}_{\mu\mu}\right)^2\right].$$

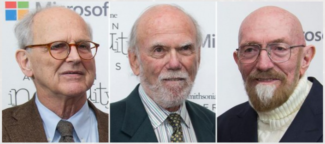
$$\dot{P} = -\frac{192\pi}{5c^5}\frac{m_1 m_2}{M^2}\left(\frac{2\pi G\,M}{P}\right)^{5/3}\underbrace{\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}}_{\text{eccentricity enhancement factor}}$$

[Peters & Mathews 1963]



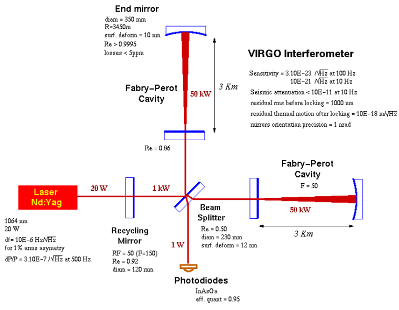
- Derivation based on flux-balance equation [Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]
- Derivation based on EoM including the radiation reaction term at 2.5PN [Damour & Deruelle 1981; Damour 1982]
- Resolution of the radiation reaction controversy [Ehlers, Rosenblum, Goldberg & Havas 1976; Will & Walker 1980]

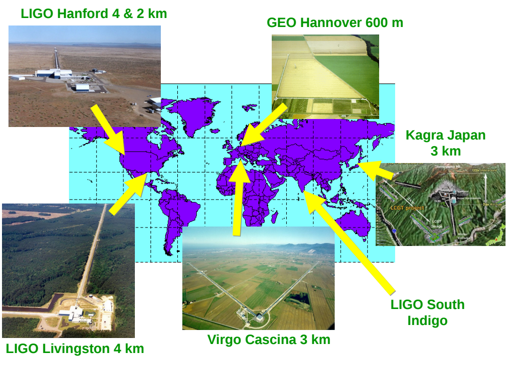
World-wide network of gravitational wave detectors



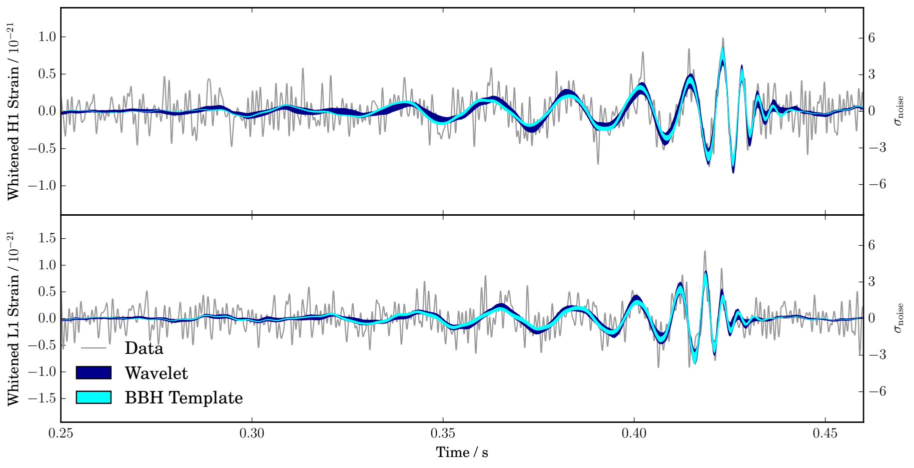
[Rainer Weiss, Barry Barish & Kip Thorne]

[Alain Brillet & Adalberto Giazotto]





Binary black-hole event GW150914 [LIGO/Virgo 2016]



The quadrupole formula works for GW150914 !

$$4\pi R^2 \mathfrak{S} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^{2} - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right].$$

- The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi}\left[\frac{256}{5}\frac{G\mathcal{M}^{5/3}}{c^5}(t_\mathrm{c} - t)\right]^{-3/8}$$

- Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96}\frac{c^5}{G\pi^{8/3}} f^{-11/3}\dot{f}\right]^{3/5}}$$

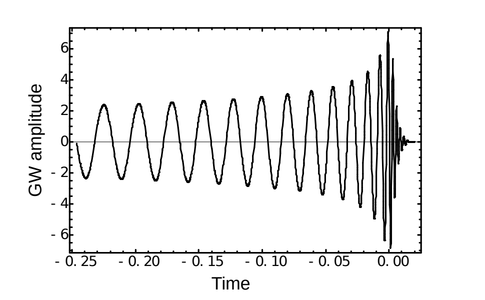
  which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

- The GW amplitude is predicted to be

$$h \sim 4.1 \times 10^{-22} \left(\frac{\mathcal{M}}{M_\odot}\right)^{5/6} \left(\frac{100\,\mathrm{Mpc}}{R}\right) \left(\frac{100\,\mathrm{Hz}}{f_\mathrm{merger}}\right)^{-1/6} \sim 1.6 \times 10^{-21}$$

- The distance $R = 400\,\mathrm{Mpc}$ is measured from the signal itself [Schutz 1986]

The gravitational chirp of compact binaries



- **Inspiralling phase**
  - Post-Newtonian theory
  - Point-particle approximation
  - Dependence on spin precession
  - Universality of the signal in GR
  - Effacing of the internal structure
    [Brillouin 1922; Damour 1982]
- **Late inspiral**
  - Post-Newtonian + Effective theory
  - Effects due to tidal interactions
  - Dependence on internal structure
  - Phenomenological models
- **Merger and post-merger**
  - Numerical relativity
  - Test of the no-hair theorem
    [Israel 1967, 1968; Carter 1971]

3.5PN: state-of-the-art on GW templates

$$\varphi_{0\text{PN}} = 1 \qquad \Longleftarrow \text{ Einstein quadrupole formula}$$

$$\varphi_{1\text{PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{1.5\text{PN}} = -10\pi$$

$$\varphi_{2\text{PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

$$\varphi^{(\ln)}_{2.5\text{PN}} = \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi$$

$$\varphi_{3\text{PN}} = \frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\text{E} - \frac{3424}{21}\ln 2 + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3$$

$$\varphi^{(\ln)}_{3\text{PN}} = -\frac{856}{21}$$

$$\varphi_{3.5\text{PN}} = \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi$$

[see Blanchet 2014 for a review]

3.5PN: state-of-the-art on GW templates

tail terms

$$\varphi_{0\text{PN}} = 1$$

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3.5PN: state-of-the-art on GW templates

tail terms

tail-of-tail terms

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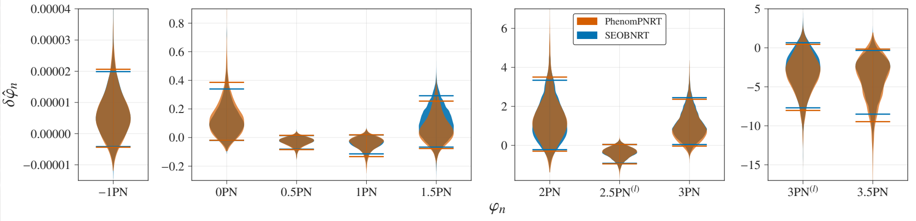
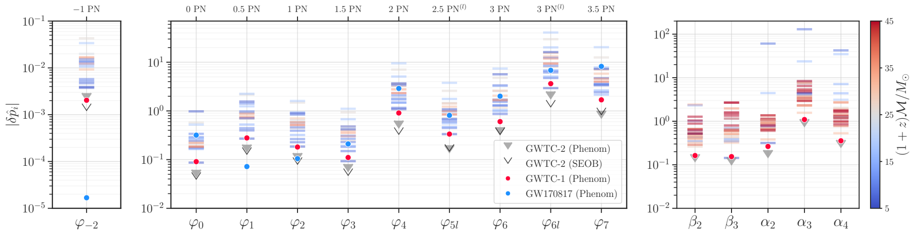
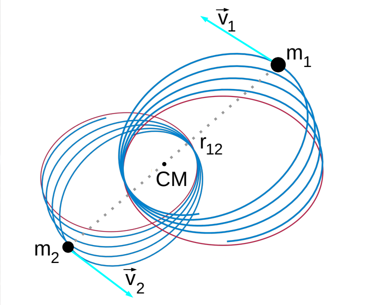
Measurement of PN parameters [LIGO/Virgo 2017-2020]



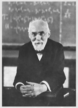


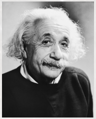
Post-Newtonian equations of motion



$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{G m_2}{r_{12}^2}\boldsymbol{n}_{12} + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \cdots\right]\boldsymbol{n}_{12} + \cdots\right\}}^{\text{1PN}}$$
$$+ \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& dissipative (tail)}}} + \mathcal{O}\left[\left(\frac{v}{c}\right)^9\right]$$

The 1PN equations of motion



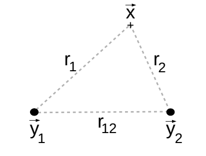




- [Lorentz & Droste 1917] The 1PN EoM and Lagrangian first derived

$$L = \frac{m_1 v_1^2}{2} + \frac{G m_1 m_2}{2 r_{12}} + \frac{1}{c^2}\Bigg\{ -\frac{G^2 m_1^2 m_2}{2 r_{12}^2} + \frac{m_1 v_1^4}{8} + \frac{G m_1 m_2}{r_{12}}\left(-\frac{1}{4}(n_{12}v_1)(n_{12}v_2) + \frac{3}{2}v_1^2 - \frac{7}{4}(v_1 v_2)\right)\Bigg\} + 1 \leftrightarrow 2 + \mathcal{O}\left(\frac{1}{c^4}\right)$$

- [Einstein, Infeld & Hoffmann 1938] The famous EIH paper
  - EoM deduced from the vacuum Einstein field equations
- [Fock 1939] Motion of the CM of extended compact objects
  - Introduces the important function for PN calculations

$$\boxed{g = \ln(r_1 + r_2 + r_{12})} \quad \text{such that} \quad \Delta g = \frac{1}{r_1 r_2}$$



4PN: state-of-the-art on equations of motion

3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EoM
	[Blanchet, Damour & Esposito-Farèse 2004]	Surface integral method
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014, 2016]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017ab]	Fokker Lagrangian
	[Foffa & Sturani 2013, 2019; Foffa, Porto, Rothstein & Sturani 2019]	Effective field theory
	[Blümlein, Maier, Marquard & Schäfer 2020]	EFT Hamiltonian

- ADM Hamiltonian: One regularization ambiguity left at 4PN order and fixed by comparison with GSF calculations
- Fokker Lagrangian: First complete derivation of the EoM at 4PN order without regularization ambiguities

MULTIPOLE MOMENTS IN d DIMENSIONS
&
4PN GRAVITATIONAL WAVE GENERATION

Based on collaborations with

Guillaume Faye, **Quentin Henry**, **François Larrouturou** & **David Trestini**

Field equations and Green's function in d dimensions

- Einstein's field equations in harmonic (de Donder) coordinates

$$\partial_\nu h^{\mu\nu} = 0 \qquad \text{(harmonic gauge condition)}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\,\tau^{\mu\nu} \qquad \text{(wave equation in } D = d+1 \text{ dimensions)}$$

$$\tau^{\mu\nu} = |g| T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu} \qquad \text{(matter + gravitation pseudo tensor)}$$

- The Green's function is implemented in the real space-time domain

$$G_{\text{ret}}(\mathbf{x}, t) = -\frac{\tilde{k}}{4\pi}\,\frac{\theta(t-r)}{r^{d-1}}\,\gamma_{\frac{1-d}{2}}\left(\frac{t}{r}\right)$$

$$\gamma_{\frac{1-d}{2}}(z) \equiv \frac{2\sqrt{\pi}}{\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2}-1)}\left(z^2-1\right)^{\frac{1-d}{2}}$$

The multipole expansion outside the matter source

- The multipole expansion $\mathcal{M}(h^{\mu\nu})$ is a retarded solution the *vacuum* field equations $\Box\mathcal{M}(h^{\mu\nu}) = \mathcal{M}(\Lambda^{\mu\nu})$ valid formally everywhere except at $r = 0$

$$\mathcal{M}(h^{\mu\nu}) = \overbrace{\mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \Big[\Big(\frac{r}{r_0}\Big)^B}^{\text{regularization when } r\to 0} \mathcal{M}(\Lambda^{\mu\nu}) \Big] \underbrace{- \frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \hat{\partial}_L \widetilde{\mathcal{F}}^{\mu\nu}_L}_{\text{retarded homogeneous solution}}$$

$$\Box\widetilde{\mathcal{F}}^{\mu\nu}_L(r,t) = 0 \quad \text{in } d \text{ dimensions}$$

- The multipole moment functions $\mathcal{F}^{\mu\nu}_L(t)$ are symmetric-trace-free (STF) with respect to their ℓ indices $L \equiv i_1 \cdots i_\ell$

$$\widetilde{\mathcal{F}}^{\mu\nu}_L(r,t) = \frac{\tilde{k}}{r^{d-2}} \int_1^{+\infty} \mathrm{d}z \, \gamma_{\frac{1-d}{2}}(z) \, \mathcal{F}^{\mu\nu}_L(t - zr)$$

The multipole expansion matched to the PN source

- Explicit matching to a general extended PN isolated source gives

$$\mathcal{F}^{\mu\nu}_L(t) = \overbrace{\underset{B=0}{\mathrm{FP}} \int \mathrm{d}^d\mathbf{x} \left(\frac{r}{r_0}\right)^B}^{\text{IR regularization}} \hat{x}_L \int_{-1}^{1} \mathrm{d}z\, \delta^{(d)}_\ell(z) \underbrace{\overline{\tau}^{\mu\nu}(\mathbf{x}, t + zr)}_{\text{PN expansion of the pseudo-tensor}}$$

$$\delta^{(d)}_\ell(z) \equiv \frac{\Gamma\left(\frac{d}{2}+\ell\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}+\ell\right)} (1-z^2)^{\frac{d-3}{2}+\ell}$$

- The $B\varepsilon$ regularization
  - first apply the limit $B \to 0$ in generic dimensions $d = 3 + \varepsilon$
  - then the usual dimensional regularization when $\varepsilon \to 0$

Mass and current irreducible multipole moments

- The irreducible decomposition of $\mathcal{F}_L^{\mu\nu}$ reads (with $\langle \cdots \rangle$ the STF projection)

$$\boxed{\begin{aligned}
\mathcal{F}_L^{00} &= R_L \\
\mathcal{F}_L^{0i} &= T_{iL}^{(+)} + T_{i|\langle i_\ell L-1\rangle}^{(0)} + \delta_{i\langle i_\ell} T_{L-1\rangle}^{(-)} \\
\mathcal{F}_L^{ij} &= U_{ijL}^{(+2)} + \underset{L}{\mathrm{STF}}\,\underset{ij}{\mathrm{STF}} \Big[U_{i|i_\ell jL-1}^{(+1)} + \delta_{ii_\ell} U_{jL-1}^{(0)} + \delta_{ii_\ell} U_{j|i_{\ell-1}L-2}^{(-1)} \\
&\qquad + \delta_{ii_\ell}\delta_{ji_{\ell-1}} U_{L-2}^{(-2)} + W_{ij|i_\ell i_{\ell-1}L-2} \Big] + \delta_{ij} V_L
\end{aligned}}$$

- The "mass-type" contributions R_L, $T_{L+1}^{(+)}$, $T_{L-1}^{(-)}$, $U_{L+2}^{(+2)}$, $U_L^{(0)}$, $U_{L-2}^{(-2)}$, V_L are STF in the ordinary sense
- The "current-type" contributions $T_{i|\langle i_\ell L-1\rangle}^{(0)}$, $U_{i|i_{\ell+1}L}^{(+1)}$, $U_{i|i_{\ell-1}L-2}^{(-1)}$ have more complicated symmetries

Mass and current irreducible multipole moments

- The mass moment I_L is given by the usual STF moment, but the generalization of the current moment involves two tensors $J_{i|L}$ and $K_{ij|L}$ having the symmetries of mixed Young tableaux

$$I_L = \begin{array}{|c|c|c|}\hline i_\ell & \dots & i_1 \\ \hline \end{array}$$

$$J_{i|L} = \begin{array}{|c|c|c|c|}\hline i_\ell & i_{\ell-1} & \dots & i_1 \\ \hline i & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad K_{ij|L} = \begin{array}{|c|c|c|c|c|}\hline i_\ell & i_{\ell-1} & i_{\ell-2} & \dots & i_1 \\ \hline j & i & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

- The tensor $K_{ij|L}$ is absent in 3 dimensions

$$\sharp(\text{components}) = \frac{(d-3)d(d-1)_{\ell-2}(2\ell+d-2)(\ell+d-1)}{2\ell(\ell+1)(\ell-2)!}$$

and plays no role with dimensional regularization

The 3PN current type quadrupole moment

After dimensional regularization and renormalization for the UV (and with Hadamard treatment which is sufficient for the IR) we obtain

$$J_{ij} = -\mu\Delta\left[A\,L^{\langle i}x^{j\rangle} + B\,\frac{Gm}{c^3}\,L^{\langle i}v^{j\rangle}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

$$\begin{aligned}
A &= 1 + \gamma\left(\frac{67}{28} - \frac{2}{7}\nu\right) + \gamma^2\left(\frac{13}{9} - \frac{4651}{252}\nu - \frac{\nu^2}{168}\right) \\
&\quad + \gamma^3\left(\frac{2301023}{415800} - \frac{214}{105}\ln\left(\frac{r}{r_0}\right)\right. \\
&\qquad \left. + \left[-\frac{243853}{9240} + \frac{123}{128}\pi^2 - 22\ln\left(\frac{r}{r'_0}\right)\right]\nu + \frac{44995}{5544}\nu^2 + \frac{599}{16632}\nu^3\right) \\
B &= \frac{188}{35}\nu\,\gamma
\end{aligned}$$

The 3.5PN gravitational-wave $(\ell, m) = (2,1)$ mode

$$h_+ - \mathrm{i} h_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} \, Y_{-2}^{\ell m}(\Theta, \Phi)$$

$$\boxed{\begin{aligned} H^{21} = \frac{\mathrm{i}}{3}\,\Delta\Big[& x^{1/2} + x^{3/2}\left(-\tfrac{17}{28} + \tfrac{5\nu}{7}\right) + x^2\left(\pi + \mathrm{i}\left[-\tfrac{1}{2} - 2\ln 2\right]\right) \\ & + x^{5/2}\left(-\tfrac{43}{126} - \tfrac{509\nu}{126} + \tfrac{79\nu^2}{168}\right) \\ & + x^3\left(\pi\left[-\tfrac{17}{28} + \tfrac{3\nu}{14}\right] + \mathrm{i}\left[\tfrac{17}{56} + \nu\left(-\tfrac{353}{28} - \tfrac{3}{7}\ln 2\right) + \tfrac{17}{14}\ln 2\right]\right) \\ & + x^{7/2}\bigg(\tfrac{15223771}{1455300} + \tfrac{\pi^2}{6} - \tfrac{214}{105}\gamma_\mathsf{E} - \tfrac{107}{105}\ln(4x) - \ln 2 - 2(\ln 2)^2 \\ & \qquad + \nu\left[-\tfrac{102119}{2376} + \tfrac{205}{128}\pi^2\right] - \tfrac{4211}{8316}\nu^2 + \tfrac{2263}{8316}\nu^3 + \mathrm{i}\pi\left[\tfrac{109}{210} - 2\ln 2\right]\bigg)\Big] \end{aligned}}$$

The 4PN mass quadrupole moment

Using dimensional regularisation for both the UV and IR the renormalized quadrupole in the case of circular orbits reads

$$I_{ij}^{\text{renorm}} = \mu \left(A\, x_{\langle i}x_{j\rangle} + B\, \frac{r^2}{c^2} v_{\langle i}v_{j\rangle} + \frac{G^2 m^2 \nu}{c^5 r}\, C\, x_{\langle i}v_{j\rangle} \right) + \mathcal{O}\left(\frac{1}{c^9}\right)$$

$$A = 1 + \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right)$$

$$+ \underbrace{\gamma^3\left(\frac{395899}{13200} - \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \left[\frac{3304319}{166320} - \frac{44}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu + \cdots\right)}_{\text{3PN terms}}$$

$$+ \underbrace{\gamma^4\left(-\frac{1023844001989}{12713500800} + \frac{31886}{2205}\ln\left(\frac{r}{r_0}\right) + \cdots\right)}_{\text{4PN terms}}$$

$$B = \frac{11}{21} - \frac{11}{7}\nu + \cdots$$

$$C = \overbrace{\frac{48}{7} + \gamma\left(-\frac{4096}{315} - \frac{24512}{945}\nu\right)}^{\text{2.5PN and 3.5PN terms}}$$

The 4PN mass quadrupole moment

- All UV divergences treated by dimensional regularization and all UV poles shown to be renormalized by appropriate shifts of the particles' worldlines ✓
- Presence at 4PN order of a non-local-in-time term associated with tail radiation mode and containing a crucial IR pole ✓
- IR divergences (poles $\propto \frac{1}{d-3}$) appear already at 3PN order but are cancelled (as well as the finite part beyond) by poles coming from "tails-of-tails" propagating in the wave zone ✓
- At 4PN order the IR poles are cancelled by more complicated "tails-of-memory" but there remains a crucial finite contribution specifically due to dimensional regularization ✓
- Finally we have obtained the finite renormalized 4PN quadrupole moment of compact binaries ready to be used for 4PN/4.5PN templates ☺

Towards 4.5PN templates

- The 4.5PN term in the phase is known and due to the 4.5PN tail-of-tail-of-tail effect [Marchand, Blanchet & Faye 2017; Messina & Nagar 2017]

$$\varphi_{\text{4.5PN}} = \Big(-\frac{93098188434443}{150214901760} + \frac{80}{3}\pi^2 + \frac{1712}{21}\gamma_\text{E} + \frac{3424}{21}\ln 2 + \Big[\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\Big]\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3\Big)\pi$$

$$\varphi^{(\ln)}_{\text{4.5PN}} = \frac{856}{21}\pi$$

tail-of-tail-of-tail terms

- However the 4PN term is only known from perturbative BH theory in the test-mass limit $\nu \to 0$ [Tagoshi & Sasaki 1994; Tanaka, Tagoshi & Sasaki 1996]

$$\varphi_{\text{4PN}} = \frac{2550713843998885153}{2214468081745920} - \frac{45245}{756}\pi^2 - \frac{9203}{126}\gamma_\text{E} - \frac{252755}{2646}\ln 2 - \frac{78975}{1568}\ln 3 + \mathcal{O}(\nu)$$

$$\varphi^{(\ln)}_{\text{4PN}} = -\frac{9203}{252} + \mathcal{O}(\nu)$$

Towards 4.5PN templates

- 4.5PN "tail-of-tail-of-tail" completed ✓
- 3PN mass octupole moment and 3PN current quadrupole moment completed (higher-order moments known) ✓
  - UV divergences treated by dimensional regularization
  - IR divergences treated by Hadamard regularization equivalent to dimensional regularization at that order
- 2PN mass dodecapole and current octupole, as well as higher-order moments are already known ✓
- Cubic interactions at 4PN order in the radiative quadrupole moment need to be completed in (computation to be done in ordinary $3d$)
  - relation between canonical and source/gauge moments ✓
  - relation between radiative and canonical moments

Still open problem: cubic interactions at 4PN order

$$U_{ij}(t) = I^{(2)}_{ij}(t) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail}}$$

$$+\frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty}\mathrm{d}\tau I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory}} + \text{inst. terms}\right\}$$

$$+\underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty}\mathrm{d}\tau I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail}}$$

$$+\frac{G^2}{c^8}\left\{\underbrace{M\int_0^{+\infty}\mathrm{d}\tau_1\int_0^{+\infty}\mathrm{d}\tau_2\, K(\tau_1,\tau_1)\, I^{(p)}_{a<i}(t-\tau_1)I^{(q)}_{j>a}(t-\tau_2)}_{\text{4PN tail-of-memory [Trestini et al. 2022 in progress]}} + \text{inst.}\right\}$$

$$+\underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty}\mathrm{d}\tau I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail}}+\cdots$$

Happy Birthday Brandon!