Brandon and black holes

Piotr T. Chruściel

University of Vienna

Paris, Carterfest, July 2022

Brandon's four most cited papers on Inspire

The Four laws of black hole mechanics #1

James M. Bardeen (Yale U.), B. Carter (Cambridge U.), S.W. Hawking (Cambridge U.) (1973)

Published in: *Commun.Math.Phys.* 31 (1973) 161-170

DOI cite claim 2,634 citations

Global structure of the Kerr family of gravitational fields #2

Brandon Carter (Cambridge U., DAMTP) (Mar, 1968)

Published in: *Phys.Rev.* 174 (1968) 1559-1571

DOI cite claim 1,394 citations

Axisymmetric Black Hole Has Only Two Degrees of Freedom #3

B. Carter (Cambridge U., Inst. of Astron.) (Feb, 1971)

Published in: *Phys.Rev.Lett.* 26 (1971) 331-333

DOI cite claim 935 citations

Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations #4

B. Carter (Cambridge U. (main)) (Dec 1, 1968)

Published in: *Commun.Math.Phys.* 10 (1968) 4, 280-310

DOI cite claim 731 citations

My further black hole favorites

138	**MR0334798 (48 #13116)** Bardeen, J. M.; Carter, B.; Hawking, S. W. The four laws of black hole mechanics. *Comm. Math. Phys.* 31 (1973), 161–170. (Reviewer: W. B. Bonnor) 83.35
111	**MR0239841 (39 #1198)** Carter, Brandon Hamilton-Jacobi and Schrödinger separable solutions of Einstein's equations. *Comm. Math. Phys.* 10 (1968), 280–310. (Reviewer: W. Israel) 83.53
101	**MR0465047 (57 #4960)** Carter, Brandon Black hole equilibrium states. *Black holes/Les astres occlus (École d'Été Phys. Théor., Les Houches, 1972)*, pp. 57–214. *Gordon and Breach, New York*, 1973. (Reviewer: Andrew King) 83.53

Killing horizons and orthogonally transitive groups in space-time #7

Brandon Carter (1969)

Published in: *J.Math.Phys.* 10 (1969) 70-81

DOI cite claim 156 citations

The commutation property of a stationary, axisymmetric system #10

Brandon Carter (Cambridge U., DAMTP) (1970)

Published in: *Commun.Math.Phys.* 17 (1970) 233-238

DOI cite claim 115 citations

$\Lambda = 0$: "Black Holes have No Hair"

The **analytic, nondegenerate,connected** classification in space-time dimension **four**; contributions by Israel, Hawking, Carter, Robinson, Bunting, Mazur, PTC-Costa Lopes, PTC-Sudarsky-Wald,

Stationary,
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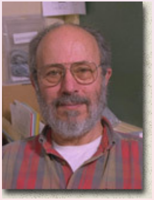

Stationary,
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in the exterior region

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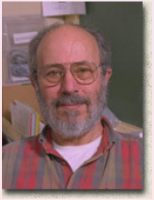

Stationary, electro-vacuum, analytic, non-degenerate, connected, regular black hole = **Kerr-Newman** in the exterior region	Static, electro-vacuum, regular black hole = **MP or RN** in the exterior region

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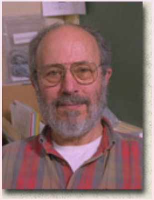
The **analytic, nondegenerate,connected** classification in space-time dimension **four**; contributions by Israel, Hawking, Carter, Robinson, Bunting, Mazur, PTC-Costa Lopes, PTC-Sudarsky-Wald, Masood-ul-Alam, Ruback, PTC-Galloway, PTC-Reall-Tod, new proof by Agostini-Mazzieri



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Half of the proof is contained in, or inspired by, Brandon's papers

How to transition from a geometric assumption

"regular black hole"

to a PDE problem

"uniqueness of solutions of a set of ODEs on a half-plane with certain singular boundary conditions"

?

This requires understanding

1. the possible groups
2. the group actions
3. the properties of Killing horizons
4. the resulting boundary conditions
5. the topology

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Diagrams?

Diagrams? Carter diagrams, Penrose diagrams, conformal diagrams ?

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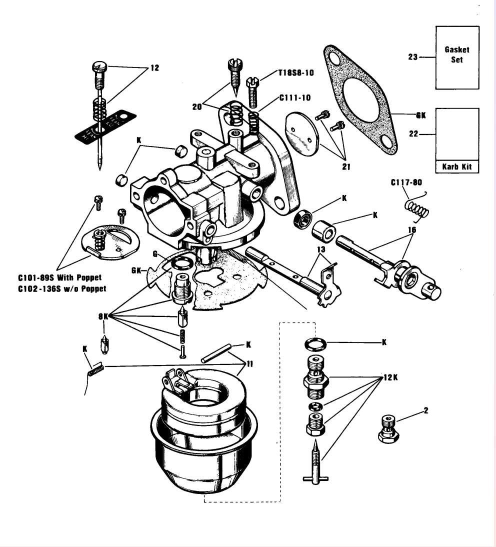

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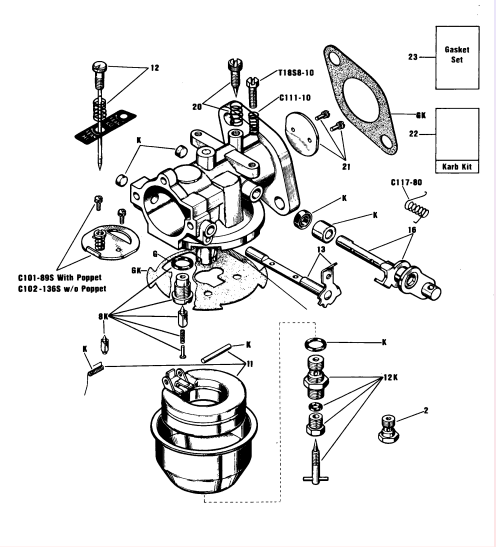
actually, carter not Carter;



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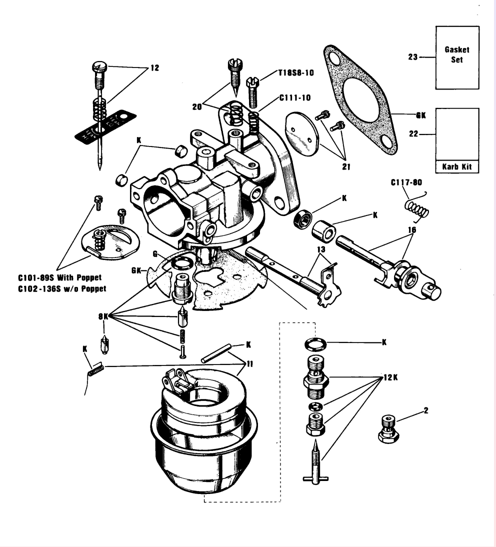

Carter Diagrams ?

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10 RAFAEL STEKOLSHCHIK

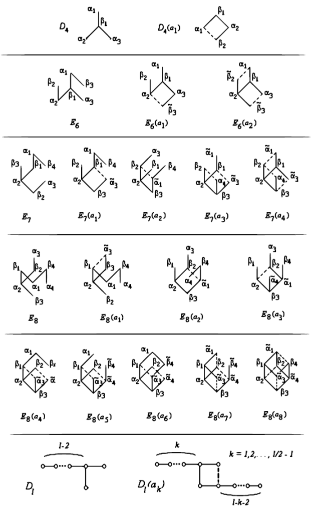
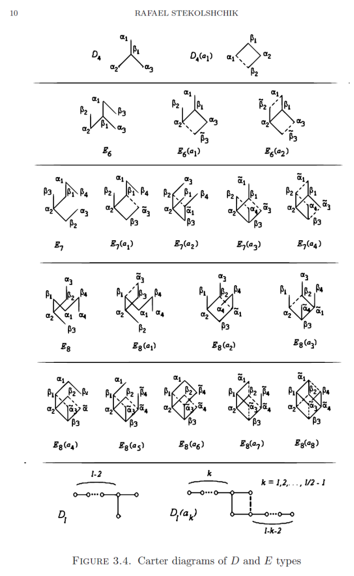


FIGURE 3.4. Carter diagrams of D and E types

Carter Diagrams ?

wrong Carter, Roger William



Carter Diagrams

visualising *causality* for general two-dimensional metrics

1. Causal-relations are conformally invariant
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Kerr metric:

$$g = -\frac{\Delta - a^2\sin^2(\theta)}{\Sigma}dt^2 - \frac{2a\sin^2(\theta)\left(r^2+a^2-\Delta\right)}{\Sigma}dtd\varphi + \frac{\sin^2(\theta)\left(\left(r^2+a^2\right)^2 - a^2\sin^2(\theta)\Delta\right)}{\Sigma}d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 ,$$

where

$$\Sigma = r^2 + a^2\cos^2\theta ,$$
$$\Delta = r^2 + a^2 - 2mr = (r-r_+)(r-r_-) ,$$
$$r_\pm = m \pm \sqrt{m^2 - a^2} .$$

Conformal diagrams - Kerr

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2-dimensional metric for construction of conformal diagrams:

1. introduce Eddington-Finkelstein coordinates,

$$dv = dt + \frac{(a^2+r^2)}{\Delta}dr\,, \qquad d\tilde\varphi = d\varphi + \frac{a}{\Delta}dr\ ,$$

2. set $\theta = const$, $\tilde\varphi = const'$,

then

$$g_2 = -\frac{F(r)}{\Sigma}dv^2 + 2dvdr\ ,$$

where

$$F(r) := r^2 + a^2\cos^2(\theta) - 2mr = (r - r_{\theta,+})(r - r_{\theta,-})\ ,$$

$$r_{\theta,\pm} = m \pm \sqrt{m^2 - a^2\cos^2(\theta)}\ .$$

Conformal diagrams - Kerr, with $\theta = 0$

B. Carter, Phys. Rev. 141, 1242 (1966)

$$g_2 = -\frac{F(r)}{\Sigma} dv^2 + 2dvdr \ ,$$

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For $\theta = 0$,

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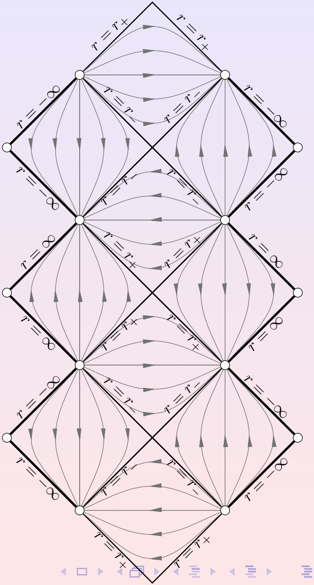

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B. Carter, Black hole equilibrium states Part I: Analytic and geometric properties of the Kerr solutions. les Houches 1973

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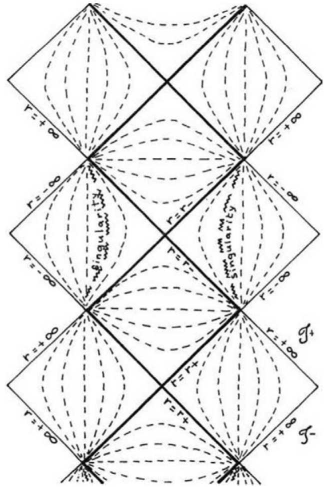

Figure 6.1. Conformal diagram of symmetry axis $\theta = 0$ of maximally extended Kerr or Kerr-Newman solution when $M^2 > a^2 + P^2 + Q^2$. In all the diagrams of this section the locus $\theta = 0$, where the axis passes through (without intersecting) the ring singularity, is marked by a broken zig-zag line.

Conformal diagrams - Kerr, with $\theta = 0$

B. Carter, Black hole equilibrium states Part I: Analytic and geometric properties of the Kerr solutions. les Houches 1973 Always read the fine print !

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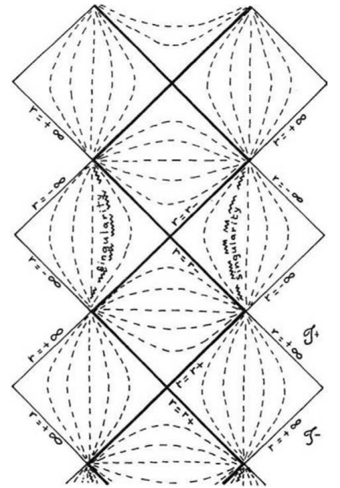

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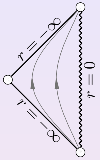
Conformal diagrams - Kerr, with $\theta = \pi/2$

$$g_2 = -\frac{F(r)}{\Sigma} dv^2 + 2dvdr ,$$

with

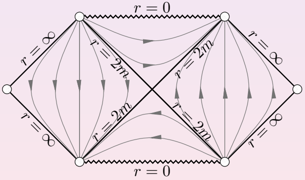
$$F(r) = r^2 + a^2\cos^2(\theta) - 2mr .$$



For $\theta = \pi/2$,

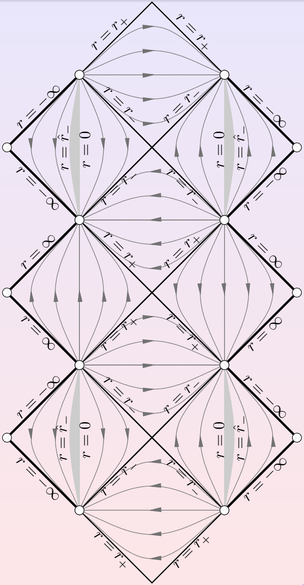
$$r_{\theta,\pm} = m \pm m ,$$

for all $m \in \mathbb{R}$.



Projection diagrams - Kerr

PTC, C. Ölz, S. Szybka, Phys.Rev.D 86 (2012) 124041, e-Print: 1211.1718 [gr-qc]



Projection diagrams - definition

PTC, C. Ölz, S. Szybka, Phys.Rev.D 86 (2012) 124041, e-Print: 1211.1718 [gr-qc]

$(\mathscr{M}, g)$... smooth space-time,
$\mathbb{R}^{1,n}$... $(n+1)$-dimensional Minkowski space-time.

A projection diagram is a pair $(\pi, \mathscr{U})$, where

$$\pi : \mathscr{M} \to \mathscr{W}$$

is a continuous map, differentiable on an open dense set, from $\mathscr{M}$ onto $\pi(\mathscr{M}) =: \mathscr{W} \subset \mathbb{R}^{1,1}$; and

$$\mathscr{U} \subset \mathscr{M}$$

is a non-empty open set, on which π is a smooth submersion, so that:

1. every smooth timelike curve $\sigma \subset \pi(\mathscr{U})$ is the projection of a smooth timelike curve γ in $(\mathscr{U}, g)$: $\sigma = \pi \circ \gamma$;
2. the image $\pi \circ \gamma$ of every smooth timelike curve $\gamma \subset \mathscr{U}$ is a timelike curve in $\mathbb{R}^{1,1}$.

Projection diagrams - Kerr;

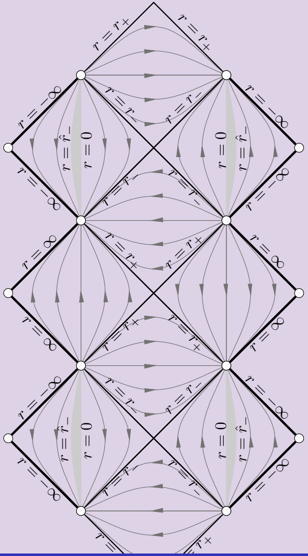
PTC, C. Ölz, S. Szybka, Phys.Rev.D 86 (2012) 124041, e-Print: 1211.1718 [gr-qc]

For $r \notin [\hat{r}_-, 0]$,

$$
\begin{aligned}
g_{proj} &= -\frac{\Delta(r^2+a^2)}{r\,(a^2(2m+r)+r^3)}dt^2 \\
&\quad +\frac{(r^2+a^2)}{\Delta}dr^2 \,, \\
\Delta &= r^2+a^2-2mr \\
&= (r-r_+)(r-r_-)\,,
\end{aligned}
$$

and assume here $m^2 > a^2$.

Carter's time machine



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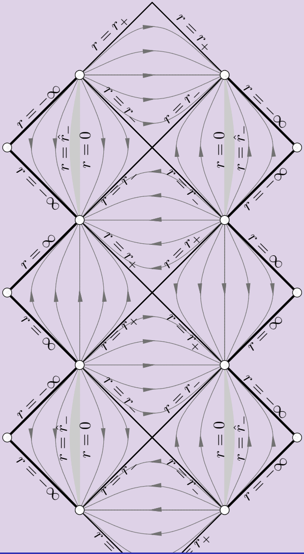

Rotating black holes with a cosmological constant

Carter-Demiański metrics (B. Carter, Comm. Math. Phys. 10 (1968), 280; M. Demiański, Acta Astronomica 23 (1973))

Kerr-(A)dS (Carter-Demiański) metric:

$$g = -\frac{\Delta - a^2\sin^2(\theta)}{\Sigma}dt^2 - \frac{2a\sin^2(\theta)\left(r^2+a^2-\Delta\right)}{\Sigma}dtd\varphi + \frac{\sin^2(\theta)\left(\left(r^2+a^2\right)^2 - a^2\sin^2(\theta)\Delta\right)}{\Sigma}d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2,$$

where

$$\Sigma = r^2 + a^2\cos^2\theta\,,$$

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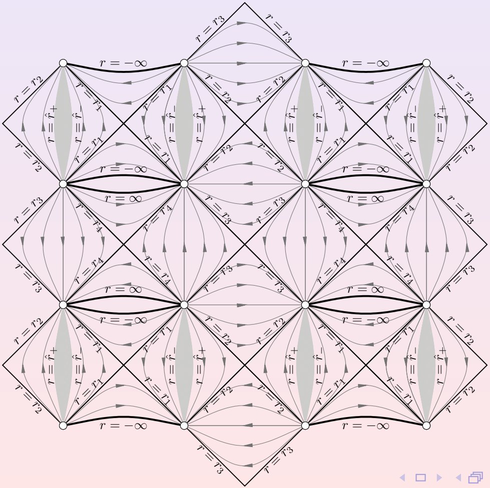
\end{aligned}
$$

Projection diagrams - Kerr-Newman-de Sitter

four simple zeros of

$$\Delta_r = \left(1 - \frac{1}{3}\Lambda r^2\right)(r^2 + a^2) - 2mr + q^2$$



Projection diagrams - Kerr-Newman-de Sitter

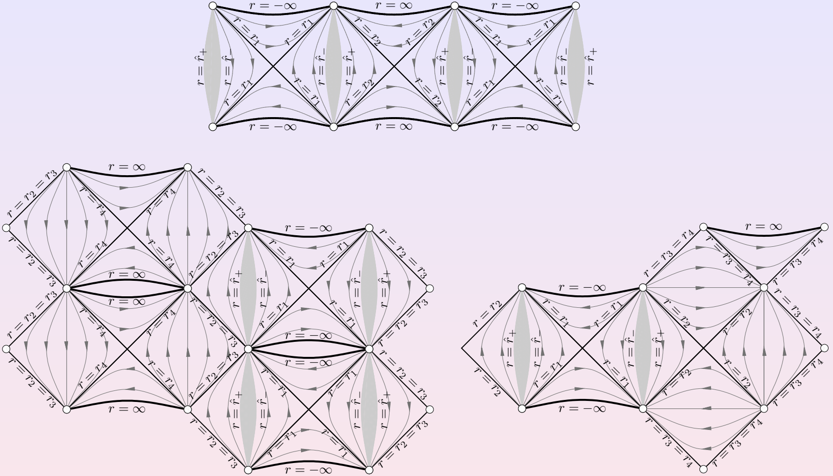


Figure: two zeros (top), or three zeros, with $r_1 < 0 < r_2 = r_3 < r_4$ (bottom left) and $r_1 < 0 < r_2 < r_3 = r_4$ (bottom right)

$$\Delta_r = \left(1 - \tfrac{1}{3}\Lambda r^2\right)(r^2 + a^2) - 2mr + q^2$$

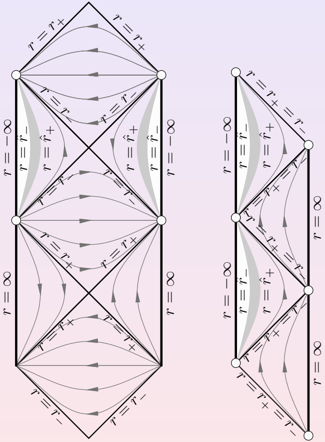


Figure: Two distinct zeros of $\Delta_r = (1 - \frac{1}{3}\Lambda r^2)(r^2 + a^2) - 2mr + q^2$ (left diagram) and one double zero (right diagram).

The structure of the ring

B. Carter, les Houches 1973

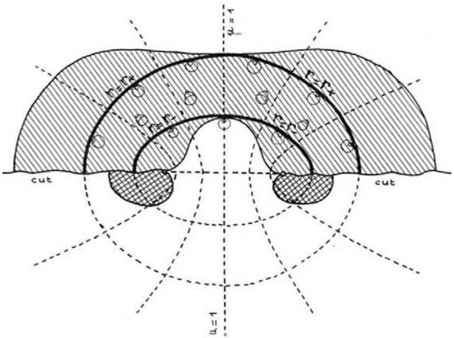


Figure 7.1. Plan of a polar 2-section on which v and $\overleftarrow{\varphi}$ are constant through maximally extended Kerr solution with $M^2 > a^2$. The ring singularity is treated as a branch point and only half of the 2-section (corresponding roughly to cos $\theta > 0$) bounded by cuts is shown – the other half should be regarded as being superimposed on the first half in the plane of the paper. The same comments apply to Figures 7.2 and 7.3. In all the diagrams of this section dotted lines are used to represent locuses on which r or θ is constant, and the positions of the Killing horizons are marked by a heavy line except for degenerate horizons which are marked by a double line. The regions in which V is negative are indicated by single shading and the regions where X is negative are marked by double shading. Some projected null cones are marked.

The structure of the ring

PTC, M. Maliborski, N. Yunes, Phys.Rev.D 101 (2020) 10, 104048, e-Print: 1912.06020 [gr-qc]

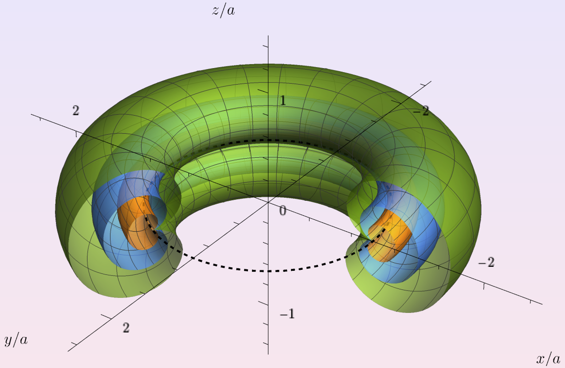
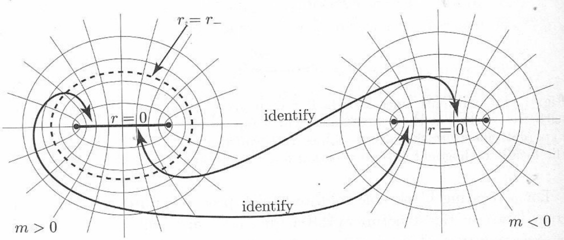


Figure: The causality-violating region at $\{t = 0\}$ in Kerr-Schild coordinates, in the negative-$\tilde{r}$ region ($a/m = .5$ corresponds to green, 1 to blue, and 2 to orange/yellow). The Killing vector ∂_φ is timelike in the region bounded by the curve and null on the boundary. The ring is located on the dotted line.

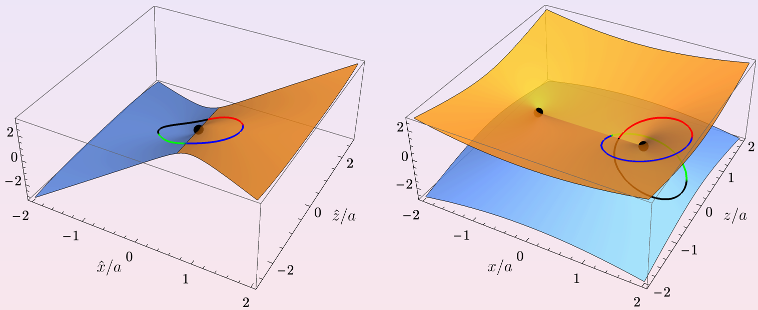
The ring and its disc

B. Carter, Les Houches 1973; figure from Griffiths & Podolsky



Reinterpretation: the minimal period of the angle around the ring is 4π

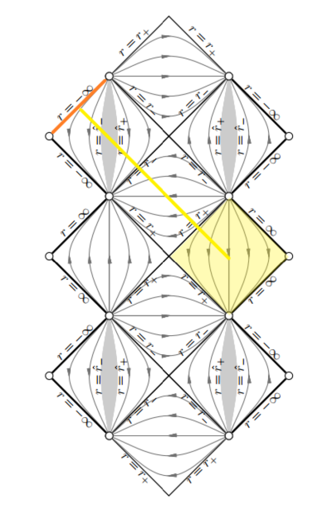
PTC, M. Maliborski, N. Yunes, Phys.Rev.D 101 (2020) 10, 104048, e-Print: 1912.06020 [gr-qc]



The ring intersects the plane spanned by the loop transversally at the dot.

Peeking through the disc

M. Maliborski, T. Sutter, PTC, https://www.quantagon.at/masters-thesis



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