Compact objects
& modified gravity

David Langlois

(APC, Paris)

Carter Fest: Black Holes and other Cosmic Systems

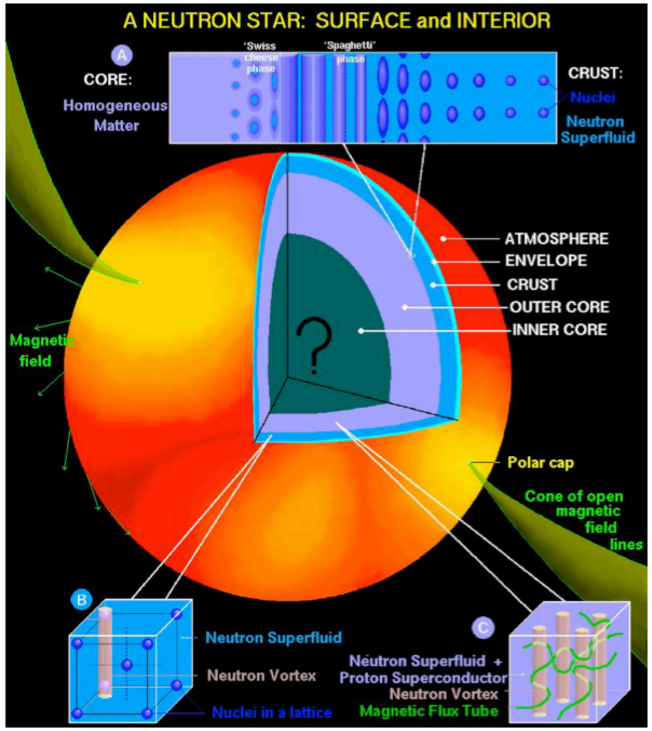
Relativistic superfluids & neutron stars with Brandon

- I met Brandon in 1991 when I started my PhD in Meudon.

- First, many discussions on history at lunch time…

- 9 papers (1994-2000) with Brandon
  Relativistic two-constituent superfluids, vortices, superconductors
  Applications to neutron stars

  – 2 papers in Phys. Rev. B (and cond-mat), with **Reinhard Prix**
  – 2 papers with David Sedrakian

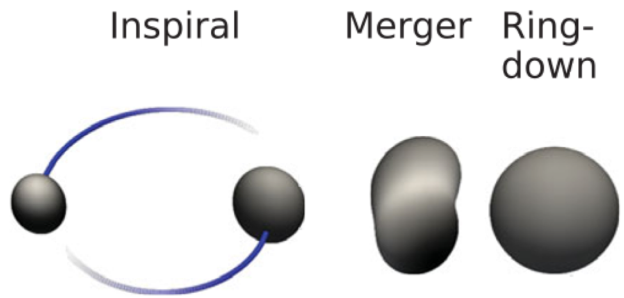


Credit: Dany Page

- From 2000, back to cosmology (braneworlds)

Black hole perturbations

- **GW astronomy** provides new windows to **test GR,** in particular in the strong field regime.



- **Ringdown phase** of a BH merger is interesting for modified gravity models; it can be described by BH linear perturbations.

- **Modified gravity**: most general framework of **scalar-tensor theories** propagating a single scalar degree of freedom **DHOST** (Degenerate Higher-Order Scalar-Tensor) theories

Based on work with Karim Noui & Hugo Roussille '21, '22

DHOST theories

- Action of **quadratic DHOST** [DL & Noui '15]

$$S=\int d^4x\sqrt{-g}\left[P(X,\phi)+Q(X,\phi)\,\Box\phi+F(X,\phi)\,R+\sum_{i=1}^{5}A_i(X,\phi)\,L_i^{(2)}\right]$$

$$L_1^{(2)}=\phi_{\mu\nu}\,\phi^{\mu\nu}\,,\quad L_2^{(2)}=(\Box\phi)^2\,,\quad L_3^{(2)}=(\Box\phi)\phi^{\mu}\phi_{\mu\nu}\phi^{\nu}$$
$$L_4^{(2)}=\phi^{\mu}\phi_{\mu\rho}\phi^{\rho\nu}\phi_{\nu}\,,\quad L_5^{(2)}=(\phi^{\mu}\phi_{\mu\nu}\phi^{\nu})^2$$

$$X\equiv\nabla_\mu\phi\nabla^\mu\phi$$
$$\phi_\mu\equiv\nabla_\mu\phi$$
$$\phi_{\mu\nu}\equiv\nabla_\nu\nabla_\mu\phi$$

The functions F and A_I satisfy three **degeneracy conditions**.

- Extension to **cubic order** (in $\phi_{\mu\nu}$) [Ben Achour et al '16]

$$L^{(3)}=F_3(X,\phi)G_{\mu\nu}\phi^{\mu\nu}+\sum_{i=1}^{10}B_i(X,\phi)L_i^{(3)}$$

- DHOST includes Horndeski, Beyond Horndeski, Einstein-scalar-Gauss-Bonnet

  e.g. (quadratic) Horndeski: $A_1=-A_2=2F_X\,,\quad A_3=A_4=A_5=0$

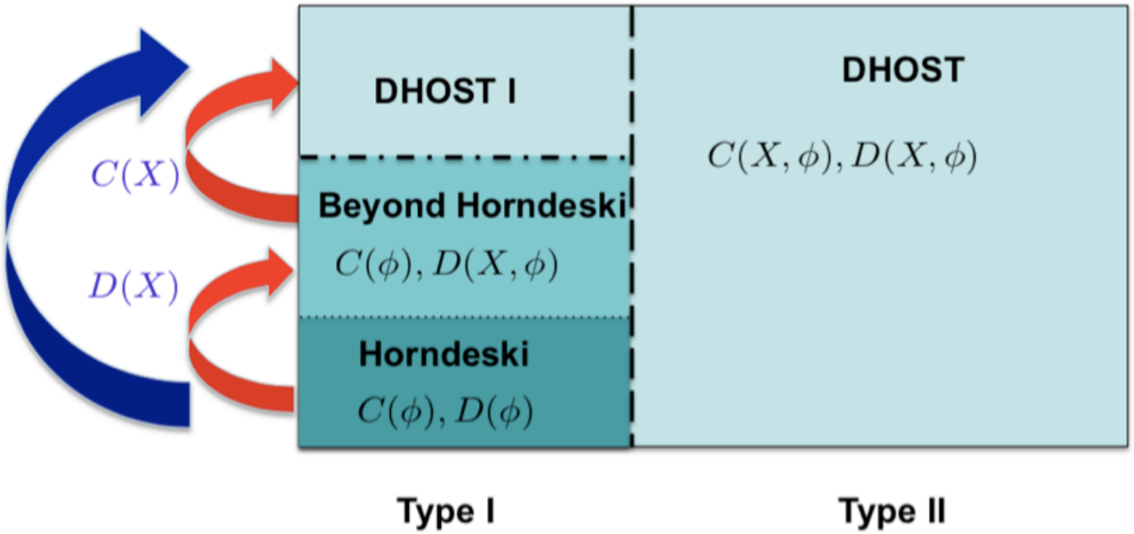
Disformal transformations

- Transformation $g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = C(X,\phi)\, g_{\mu\nu} + D(X,\phi)\, \partial_\mu\phi\, \partial_\nu\phi$

- From an action $\tilde{S}\,[\phi, \tilde{g}_{\mu\nu}]$, one gets the new action

$$S[\phi, g_{\mu\nu}] \equiv \tilde{S}\,[\phi, \tilde{g}_{\mu\nu} = C\, g_{\mu\nu} + D\, \phi_\mu \phi_\nu]$$

- DHOST families are **closed** under these transformations



- When **standard fields** are (minimally) included, two disformally related theories are **physically inequivalent** !

$$S[g_{\mu\nu}, \phi] + S_m[\Psi_m, g_{\mu\nu}] \quad \neq \quad \tilde{S}[\tilde{g}_{\mu\nu}, \phi] + S_m[\Psi_m, \tilde{g}_{\mu\nu}]$$

BH background solution

- Static spherically symmetric BH with a **nontrivial** scalar field
  - Metric: $$ds^2 = -\mathcal{A}(r)dt^2 + \frac{dr^2}{\mathcal{B}(r)} + \mathcal{C}(r)(d\theta^2 + \sin^2\theta\, d\varphi^2)$$
  - Scalar field: $$\phi(t,r) = q\,t + \psi(r)$$ [Babichev & Charmousis '13]

  [$q \neq 0$ possible in shift-symmetric theories]

- **Examples:**
  - **« stealth » Schwarzschild:** $$\mathcal{A} = \mathcal{B} = 1 - \frac{\mu}{r}$$
  - **« BCL »** [Babichev, Charmousis & Lehébel '17] $$\mathcal{A} = \mathcal{B} = 1 - \frac{\mu}{r} - \xi\frac{\mu^2}{2r^2}$$
  - **« 4d Gauss-Bonnet »** [Lu & Pang '20] $$\mathcal{A} = \mathcal{B} = 1 - \frac{2\mu/r}{1 + \sqrt{1 + 4\alpha\mu/r^3}}$$
  - **Scalar-Gauss-Bonnet** [Julié & Berti '19] $$\mathcal{A} = \mathcal{B} = 1 - \frac{\mu}{r} + a_2(r)\varepsilon^2 + \dots$$

Black hole perturbations

- In the frequency domain: $$f(t,r) = f(r)\, e^{-i\omega t}$$

- **Axial** (or odd) modes: $h_0(r), h_1(r)$ [Regge-Wheeler gauge]

$$h_{\mu\nu} = \sum_{\ell,m} \begin{pmatrix} 0 & 0 & \frac{1}{\sin\theta} h_0^{\ell m} \partial_\varphi & -\sin\theta h_0^{\ell m} \partial_\theta \\ 0 & 0 & \frac{1}{\sin\theta} h_1^{\ell m} \partial_\varphi & -\sin\theta h_1^{\ell m} \partial_\theta \\ \text{sym} & \text{sym} & 0 & 0 \\ \text{sym} & \text{sym} & 0 & 0 \end{pmatrix} Y_{\ell m}(\theta,\varphi)$$

- **Polar** (or even) modes: H_0, H_1, H_2, K (and $\delta\phi$)

$$h_{\mu\nu} = \sum_{\ell,m} \begin{pmatrix} A(r) H_0^{\ell m}(r) & H_1^{\ell m}(r) & 0 & 0 \\ H_1^{\ell m}(r) & A^{-1}(r) H_2^{\ell m}(r) & 0 & 0 \\ 0 & 0 & K^{\ell m}(r)\, r^2 & 0 \\ 0 & 0 & 0 & K^{\ell m}(r)\, r^2 \sin^2\theta \end{pmatrix} Y_{\ell m}(\theta,\varphi)$$

Axial modes in GR

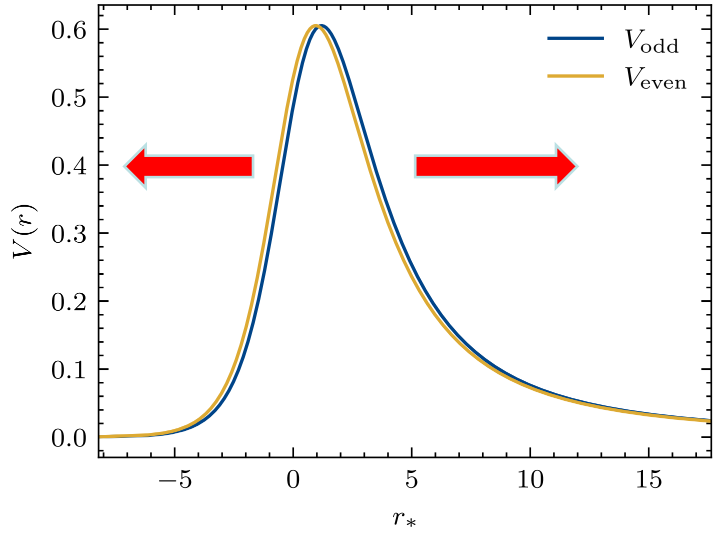
- The linearised metric eqs yield only 2 independent eqs

$$\frac{dY}{dr} = M(r)\,Y(r)\,, \quad Y = \begin{pmatrix} h_0 \\ h_1/\omega \end{pmatrix}$$

or, in a **Schroedinger** form, [Regge & Wheeler '57]

$$\frac{d^2\hat{Y}}{dr_*^2} + \left(\omega^2 - V(r)\right)\hat{Y} = 0$$

[r_* tortoise coordinate]



- **Asymptotically** ($r_* \to -\infty, +\infty$)

$$e^{-i\omega t}\,\hat{Y}(r) \approx a_+\,e^{-i\omega(t-r_*)} + a_-\,e^{-i\omega(t+r_*)}$$

outgoing ingoing

- **Quasi-normal modes**: $a_+^{\rm hor} = 0 \quad \text{and} \quad a_-^{\infty} = 0$

Axial modes in DHOST

- The equations have a similar structure:

$$\frac{dY}{dr} = MY\,, \quad M \equiv \begin{pmatrix} 2/r + i\omega\Psi & -i\omega^2 + 2i\lambda\Phi/r^2 \\ -i\Gamma & \Delta + i\omega\Psi \end{pmatrix}$$

$$\lambda \equiv \frac{\ell(\ell+1)}{2} - 1$$

where Ψ, Φ, Γ and Δ depend on the Lagrangian's functions and on the background.

$$\mathcal{F} = \mathcal{A}F_2 - (q^2 + \mathcal{A}X)A_1 - \frac{1}{2}\mathcal{A}\mathcal{B}\psi' X' F_{3X} - \frac{1}{2}\mathcal{B}\psi'(\mathcal{A}X)' B_2 - \frac{\mathcal{A}}{2\mathcal{B}}(\mathcal{B}\psi')^3 X' B_6\,,$$

$$\mathcal{F}\Psi = q\left[\psi' A_1 + \frac{1}{2}\left(\mathcal{B}\psi'^2\right)' F_{3X} + \frac{1}{2}\frac{(\mathcal{A}X)'}{\mathcal{A}}B_2 + \frac{1}{4}\left(\mathcal{B}^2\psi'^4\right)' B_6\right]\,,$$

$$\frac{\mathcal{F}}{\Phi} = F_2 - XA_1 - \frac{1}{2}\mathcal{B}\psi' X' F_{3X} - \frac{1}{2}\mathcal{B}\psi' \frac{(\mathcal{C}X)'}{\mathcal{C}} B_2 - \frac{1}{2}\mathcal{B}\psi' X X' B_6\,,$$

$$\Gamma = \Psi^2 + \frac{1}{2\mathcal{A}\mathcal{B}\mathcal{F}}\left(2q^2 A_1 + 2\mathcal{A}F_2 + \mathcal{A}\mathcal{B}\psi' X' F_{3X} + q^2\frac{(\mathcal{A}X)'}{\mathcal{A}\psi'}B_2 + q^2\mathcal{B}\psi' X' B_6\right)\,,$$

$$\Delta = -\frac{\mathcal{F}'}{\mathcal{F}} - \frac{\mathcal{B}'}{2\mathcal{B}} + \frac{\mathcal{A}'}{2\mathcal{A}}$$

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$$\lambda \equiv \frac{\ell(\ell+1)}{2} - 1$$

where Ψ, Φ, Γ and Δ depend on the Lagrangian's functions and on the background.

- After time redefinition, one can get a **Schroedinger-like** equation

$$\frac{d^2\mathcal{Y}}{dr_*^2} + \left(\frac{\omega^2}{c_*^2(r)} - V(r)\right)\mathcal{Y} = 0 \qquad \frac{dr}{dr_*} \equiv n(r)$$

where $c_*(r)$ and $V(r)$ depend on the choice $n(r)$.

Effective metric for axial modes

- **Correspondence**

DHOST axial modes in $g_{\mu\nu}$	$\longleftrightarrow$	**GR** axial modes in $\tilde{g}_{\mu\nu}$

with the **effective metric**

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = |\mathcal{F}| \sqrt{\frac{\Gamma \mathcal{B}}{\mathcal{A}}} \left(-\Phi(dt - \Psi dr)^2 + \Gamma \Phi \, dr^2 + \mathcal{C} \, d\Omega^2\right)$$

- **Quadratic** DHOST theories

The disformal transformation such that $\hat{F} = 1$ and $\hat{A}_1 = 0$ yields

$$\hat{g}_{\mu\nu} = \sqrt{F(F - XA_1)} \left(g_{\mu\nu} + \frac{A_1}{F - XA_1} \phi_\mu \phi_\nu \right)$$

which coincides with $\tilde{g}_{\mu\nu}$

Example: stealth Schwarzschild

- Coefficients

$$\Psi = \frac{\zeta\, r_s^{1/2} r^{3/2}}{(r - r_s)(r - r_g)}\,, \quad \Phi = \frac{r - r_g}{(1+\zeta) r}\,, \quad \Gamma = \frac{(1+\zeta) r^2}{(r - r_g)^2}\,, \quad \Delta = \frac{1}{r} - \frac{1}{r - r_g}$$

$$\zeta \equiv 2q^2\alpha\,, \qquad r_g \equiv (1+\zeta) r_s \qquad\qquad [\,\zeta = 0 :\ \ \mathrm{GR}\,]$$

- Effective metric: **Schwarzschild** with a **displaced horizon**

$$d\tilde{s}^2 \;=\; -\left(1 - \frac{R_g}{R}\right) dT^2 + \left(1 - \frac{R_g}{R}\right)^{-1} dR^2 + R^2 d\Omega^2 \qquad \left[R = (1+\zeta)^{1/4} r\right]$$

- Potential [with $c(r) = 1$]

$$V_{c=1}(r) = \left(1 - \frac{r_g}{r}\right) \frac{\ell(\ell+1) r - 3 r_g}{(1+\zeta)\, r^3}$$

[see also Tomikawa & Kobayashi '21]

Same potential as in GR, but with r_g instead of r_s (and a rescaling).

Other effective metrics

- BCL solution: $\mathcal{A} = \mathcal{B} = 1 - \frac{\mu}{r} - \xi \frac{\mu^2}{2r^2}$

$$d\tilde{s}^2 \;=\; \sqrt{1 + \xi \frac{\mu^2}{r^2}} \left[-\mathcal{A}(r) dt^2 + \frac{1}{\mathcal{A}(r)} \left(1 + \xi \frac{\mu^2}{r^2} \right) dr^2 + r^2 d\Omega^2 \right]$$

BH geometry with the same horizon

- 4d Gauss-Bonnet solution: $\mathcal{A} = \mathcal{B} = 1 - \frac{2\mu/r}{1 + \sqrt{1 + 4\alpha\mu/r^3}}$

$$d\tilde{s}^2 \;\simeq\; -c_1 (z-1)^{1/4} dt^2 + \frac{c_2}{(z-1)^{5/4}} dz^2 + \frac{c_3}{(z-1)^{1/4}} d\Omega^2$$

$$[z \equiv r/r_+]$$

Naked singularity

Polar modes

- The linearised metric equations yield
  - 2 independent equations in GR (1 dof)
  - **4 independent equations** in DHOST theories (2 dof)

- In **GR**: 2-dimensional system $Y' = M\,Y$, which can be written in a Schroedinger form. [Zerilli '70]

- In **DHOST**, the system $Y' = M\,Y$ is now 4-dimensional, with

$$Y = {}^{T}(K\ \delta\phi\ H_1\ H_0)$$

- It is convenient to do an **asymptotic analysis** of the first-order system.

Asymptotics of a diffential system

- Instead of a Schroedinger-like approach, one can use directly the initial first-order equations of motion and their asymptotic limit:

$$\frac{dY}{dz} = M(z)\,Y\,, \quad M(z) = M_r z^r + M_{r-1} z^{r-1} + \ldots \quad (z \to \infty)$$

- The generic solution is of the form

$$Y(z) = e^{\boldsymbol{\Upsilon}(z)}\, z^{\boldsymbol{\Delta}}\, \mathbf{F}(z)\, Y_0, \qquad (z \to \infty)$$

- There exists a well-defined algorithm to determine the diagonal matrices $\boldsymbol{\Upsilon}(z)$ and $\boldsymbol{\Delta}$. [Balser '99]

Idea: diagonalise, order by order, the matrix *M*, with $Y(z) = P(z)\,\tilde{Y}(z)$

$$\frac{d\tilde{Y}}{dz} = \tilde{M}(z)\,\tilde{Y}, \qquad \tilde{M}(z) \equiv P^{-1} M P - P^{-1}\frac{dP}{dz}$$

Polar modes

- Study the **asymptotic behaviour** of the 4-dim system at spatial infinity and near the horizon, and **extract the asymptotic independent modes**.

- At spatial infinity, one can identify
  - 2 « gravitational » modes
  - 2 « scalar» modes

- Similar results near the horizon

- Well-behaved asymptotic « scalar » modes for EsGB, although not for stealth Schwarzschild, BCL and 4d-GB.

Conclusions

- Analysis of the **BH linear perturbations** in **DHOST** theories
- **Axial modes**: Correspondence between **DHOST axial** modes and **GR axial** modes in an **effective metric.**
- **Polar modes:** the structure is much more complicated than in GR (4-dim system).
- **Systematic approach** to disentangle the modes **asymptotically**. Also useful to get the boundary conditions for numerical integration.

Happy birthday Brandon !