# Two discs and a missing triangle: the maximally extended Kerr black hole revisited

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# The Kerr solution

Kerr published his solution in 1963, using an approach prompted by that of the well-known NUT paper of the same year. He gave the solution in its Kerr-Schild form (*) $g_{ab} = \eta_{ab} - 2Sk_ak_b$, where $\eta_{ab}$ is the Minkowski metric, and $k_a$ is null in both metrics $\eta_{ab}$ and $g_{ab}$, as

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2 + \frac{2mr^3}{r^4 + a^2z^2}(k_i \mathrm{d}x^i)^2 \qquad (1)$$

where the vector $\boldsymbol{k}$ and the variable $r$ are given by rather lengthy expressions in $(x, y, z)$ which I will skip over here. This is a stationary axisymmetric asymptotically flat solution for which Kerr identified the parameters $m$ and $a$ as the mass and angular momentum per unit mass.

(*) For more on such metrics see Chapter 32 of the exact solutions book: Stephani et al (2003).

# The Kerr black hole

Subsequent papers have added electromagnetic field parameters, a NUT parameter, and the cosmological constant $\Lambda$, giving the Kerr-Newman-NUT-(anti) de Sitter (KNN(A)S) metric. As Piotr Chruściel's talk illustrated, the analytic extensions for the KNN(A)S cases are very different, and for simplicity today I shall speak only about the black hole variant of the vacuum solution, so $m$ and $a$ are the only parameters and $a^2 < m^2$ and $m > 0$.

Even in this case, and with two ignorable coordinates, the geometry of the Kerr black hole has been tricky to understand and apply. See for example the reviews in "The Kerr Spacetime: Rotating Black Holes in General Relativity" ed. D. L. Wiltshire, M. Visser, and S. M. Scott. Cambridge University Press (2009). Partly for that reason a variety of coordinates have been used.

# Boyer-Lindquist coordinates

The most frequently used coordinates were introduced by Boyer and Lindquist (B-L) in 1967. In them, the Kerr metric reads

$$\mathrm{d}s^2 = -\frac{Q}{R^2}\left[\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right]^2 + \frac{R^2}{Q}\mathrm{d}r^2 + \frac{\sin^2\theta}{R^2}\left[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi\right]^2 + R^2\mathrm{d}\theta^2 \tag{2}$$

where

$$R^2 = r^2 + a^2\cos^2\theta \tag{3}$$

$$Q = r^2 - 2mr + a^2. \tag{4}$$

Here the variable ranges are $-\infty < t < \infty$, $-\infty < r < \infty$, and $0 < \theta < \pi$, while $\phi$ is periodic with period $2\pi$.

# Horizons and stationary limits

In B-L coordinates, $r$ is spacelike where $Q > 0$ and timelike where $Q < 0$. The boundaries between these regions are at $r_\pm := m \pm \sqrt{m^2 - a^2}$, the two horizons. Similarly, $g_{tt}$ changes sign at the stationary limit surfaces $r_{e^\pm} := m \pm \sqrt{m^2 - a^2 \cos^2\theta}$, the boundaries of the ergoregion within which curves with fixed $r$ and $\theta$ must orbit in the same sense as the black hole rotates.. Note that $r_{e^-} \leq r_- < r_+ < r_{e^+}$.

One can, following work of Abdelqader and Lake (2013) and of Page and Shoom (2015) as clarified and improved in Brooks et al (2018), find invariants (for the KNN(A)S case) which vanish where these surfaces occur: some may also vanish at other curves or surfaces. (See Chruściel et al (2020) for where the Kretschmann scalar vanishes.)

# Carter's paper of 1966

In this paper, Brandon studied the analytic extension of the 2-dimensional metric on the symmetry axis $\theta = 0$, in coordinates in which it takes the form

$$\mathrm{d}s^2 = -[1 - 2mr/(r^2 + a^2)]\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r. \quad (5)$$

Integrating the null geodesics in this space he showed that they are incomplete as $r \to r_+$ and $u \to -\infty$ and as $r \to r_-$ and $u \to \infty$.

He then found the maximal analytic extension of this surface across $r = r_\pm$. Inspired by work of Penrose in 1964, he illustrated the result in a conformal diagram, in which the infinities in $r$ are mapped to finite ranges. (Such diagrams are often now called Carter-Penrose diagrams.) It turned out the same diagram applies for any $\theta \neq \pi/2$.

# Brandon's conformal diagram

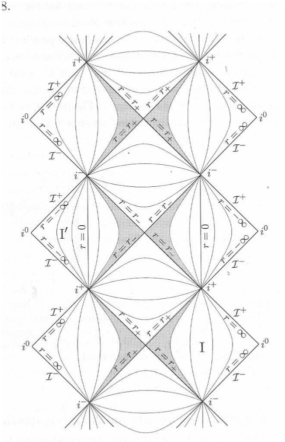



Figure: 1. The conformal diagram of the maximally extended Kerr black hole for a worldsheet on which $\theta(\neq \pi/2)$ and $\phi$ are constant. The shaded areas indicate the ergoregions. Figure taken from Griffiths and Podolsky(2009), Fig. 11.8, which is itself after Carter (1966). The region I' and those above and to its right form a B-L coordinate patch.

# The ring singularity

Kerr had noted that his solution is analytic everywhere except (in B-L coordinates) at $r = 0$ and $\theta = \pi/2$. The only nonzero component of the curvature, which in Newman-Penrose notation is $\Psi_2 = -m/(r + ia\cos\theta)^3$, is singular at those values. This is the now well-known "ring singularity". (Note by the way that Kerr's 2020 Oscar Klein lecture had the provocative title "Kerr black holes have no singularities".)

The detailed geometry near the singularity was studied by Chruściel et al (2020). I note that in Piotr's talk he showed how using toroidal coordinates there can clarify the structure there, including the fact noted by Brandon in his 1968 paper discussing the 4-dimensional case, that the Killing vector $\partial_\phi$ becomes timelike in a region there and so causality is lost.

# The disc(s) spanning the singularity

Boyer and Lindquist discussed the appearance of the surfaces $r =$ constant taking the $(x, y, z)$ of Kerr's coordinates to be Cartesian. These surfaces are confocal ellipsoids for $r > 0$, of which they say that at $r = 0$ "the ellipsoid degenerates to a disk". They similarly showed that surfaces of constant $\theta > 0$ are half-hyperboloids of one sheet, confocal to the ellipsoids. The 'half' is because they implicitly assume $r \geq 0$, so the surfaces terminate on $r = 0$.

This argument only shows that the region $r = 0$ is homeomorphic to a disc, not necessarily isometric to a flat disc. However, the induced metric on $r = 0$, $t =$ constant, is

$$\mathrm{d}s^2 = a^2 \sin^2\theta \mathrm{d}\phi^2 + a^2 \cos^2\theta \mathrm{d}\theta^2 \quad (6)$$

which, defining $\chi = a \sin\theta$, can be written as $\mathrm{d}\chi^2 + \chi^2 \mathrm{d}\phi^2$, the usual metric of a flat disc in polar coordinates. (García-Compeán and Manko have shown that in the generalizations of the Kerr black hole this region is no longer flat, though it still has disc topology.)

# Two discs rather than one

While this conclusion is fine as far as it goes, I want to argue that it is preferable to think of $r = 0$ as two discs, not one. This is because taking the range $\theta \in (0, \pi)$ (excluding $\theta = \pi/2$) covers all values of $|\chi| \in (0, a)$ twice, once from $0 < \theta < \pi/2$ and once from $\pi/2 < \theta < \pi$. Both these discs span the ring singularity. Taking this viewpoint makes some aspects of the analytic extension of the Kerr metric easier to understand.

The presence of two discs can be seen in, for example, Figure 2 of García-Compeán and Manko (2014) and Figure 11.2 of Griffiths and Podolsky (2009), and, implicitly, the "double cover" description in Piotr Chruściel's talk. [García-Compeán and Manko also argue that the discs are really cones, with a cusp at the centre, but I do not think their argument for that is correct.] It is of course tempting to identify the two discs.

# $r < 0$ and $m < 0$ in Boyer-Lindquist coordinates

Several authors have noted that the region $r < 0$ of the $m > 0$ metric in B-L form also gives the $r > 0$ region of an $m < 0$ solution, because the metric coefficients only involve the product $mr$. This has led to the description of the analytic extension of the region $r > 0$ through the disc(s) illustrated in this diagram:

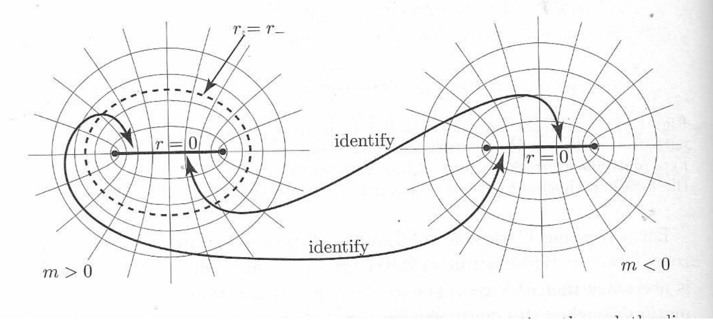



Figure: 2. This is taken from Figure 11.2 of Griffiths and Podolsky (2009). See also Brandon's 1973 lectures.

# Comments on this representation

While of course perfectly correct as a representation of the whole region covered by a set of B-L coordinates, I feel it has led to some authors thinking of it as a gluing of two distinct spacetimes, one with $m > 0$ and one with $m < 0$, whereas these are really just different regions of one coordinate chart for the $m > 0$ case. (Here I am glossing over whether the singularity can properly be considered to be at points in a chart neighbourhood.)

I also note that in the book of Griffiths and Podolsky it is, perhaps confusingly, captioned as depicting "the maximal analytic extension... through the disc at $r = 0$", which is correct but could be misleading: it is not the same as the full maximal extension which needs the extensions through the horizons found by Brandon.

# The conformal diagram for the equatorial plane

There is a well-known conformal diagram for the maximal analytic extension of $\theta = \pi/2$, $\phi$ constant.

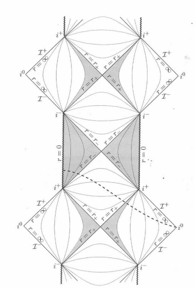
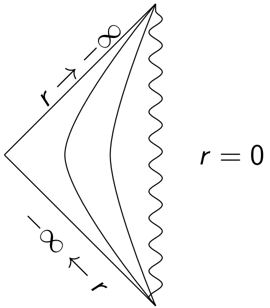


Figure: 3. Maximal analytic extension of $\theta = \pi/2$, $\phi$ constant. This figure is taken from Figure 11.7 of Griffiths and Podolsky(2009).

However, the caption should add the condition $r \geq 0$. Then it is correct but does not represent all the points with $\theta = \pi/2$ in the maximal analytic extension of the Kerr solution.

# The missing triangle

To achieve that one must add (multiple copies of) a triangle to the previous conformal diagram. The extra points are those for $r < 0$, $\theta = \pi/2$ at the chosen constant $\phi$ which cannot be reached by analytic continuation from $r \geq 0$, $\theta = \pi/2$ within that two-dimensional space, but can be reached in the full analytic continuation. The triangle is the very simple picture for the extra points below.



# Bringing the two discs and the triangle together

To better illustrate the argument one can look at the following diagram (for a KN solution with specific values) sent to me by an anonymous referee (used with the originator's permission).

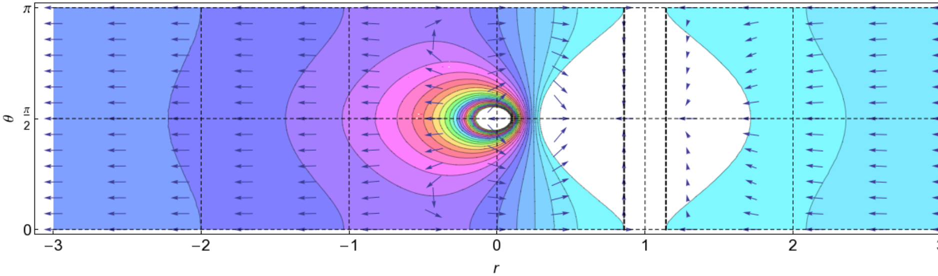



Figure: 4. A representation of the $(r, \theta)$ plane in B-L coordinates (for $|r| \leq 3$) for a (Kerr-Newman) solution with $m = 1$. The white area on the right indicates the stationary limits and the region between the horizons. Note that the equivalent diagram for $m = -1$ is the same as this but reversed left-right.

# The amended figure for $\theta = \pi/2$, $\phi$ constant

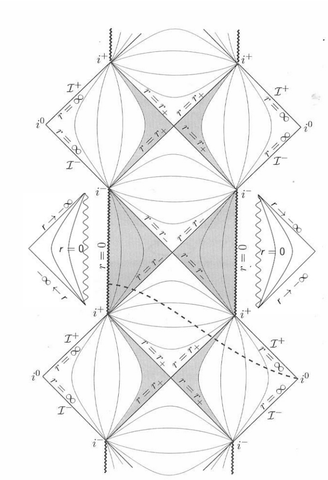



Figure: 5. Same as Figure 3 with two of the extra triangles shown

# The maximal analytic extension of the Kerr black hole

For a conformal three-dimensional depiction of the maximal extension we can now stack for each $\theta \neq \pi/2$ in a B-L chart, i.e. for each line $\theta =$ constant $\neq \pi/2$ in Figure 4, the two-dimensional figure found by Brandon (Fig. 1 above) together with a single copy of Figure 5 for the line $\theta = \pi/2$ in Figure 4.