Symmetry operators and separation of variables for the Dirac equation

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Outline

Hamilton-Jacobi, Helmholtz, and Dirac equations

We consider the following equations defined on a n-dimensional spin manifold M with contravariant metric tensor $g^{\mu\nu}$ and covariant derivative ∇_μ:

The Hamilton-Jacobi (HJ) equation for the geodesics:

$$g^{\mu\nu}\partial_\mu W \partial_\nu W = m. \tag{1}$$

The Klein-Gordon (KG) equation

$$\mathbb{H}\psi := g^{\mu\nu}\nabla_\mu\nabla_\nu\psi - m^2\psi = 0. \tag{2}$$

The Dirac (D) equation

$$\mathbb{H}\psi := i\gamma^a\nabla_a\psi - m\psi = 0, \tag{3}$$

where the γ^a satisfy

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\mathbb{I},$$

and η^{ab} denotes the (constant) frame metric.

Dirac equation

In (3)

$$\nabla_a := e_a^\mu \nabla_\mu,$$

denotes the frame covariant derivative associated to the spin frame e_a^μ.
The covariant derivative of a spinor ψ is defined by

$$\nabla_\mu \psi = \partial_\mu \psi + \tfrac{1}{8} \Gamma_\mu^{ab} [\gamma_a, \gamma_b] \psi,$$

where the spin connection is given by

$$\Gamma_\mu^{ab} = e_\alpha^a (\Gamma_{\beta\mu}^\alpha e^{b\beta} + \partial_\mu e^{b\alpha}). \tag{4}$$

In (4) $\Gamma_{\beta\mu}^\alpha$ denotes the the Levi-Civita connection of the metric

$$g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b$$

induced by the spin frame.

Separation of Variables: HJ and KG equations

Hamilton-Jacobi equation: sum separability ansatz.

The HJ equation is separable if there exists a separable *complete integral* of the form

$$W(\boldsymbol{x}, \boldsymbol{c}) = \sum_{i=1}^{n} W_i(q^i, \boldsymbol{c}),$$

where $\boldsymbol{c} = (c_1, \ldots, c_n)$ and $\det\left(\frac{\partial^2 W}{\partial c_i \partial q^j}\right) \neq 0$.

Klein-Gordon equation: product separability ansatz.

The KG equation is separable if there exists a separable solution (satisfying some completeness condition) of the form

$$\psi(\boldsymbol{x}, \boldsymbol{c}) = \prod_{i=1}^{n} \psi_i(x^i, \boldsymbol{c}), \tag{5}$$

such the functions ψ_i are solutions of a set of ODEs in the variables x^i.

Separation of variables: D equation

Dirac equation: local product separability ansatz.

The Dirac equation is said to be separable in a spin frame e_a^μ with respect to local coordinates $\boldsymbol{x} := (x^1, \ldots, x^n)$ if there exists a separable solution of the form

$$\psi(\boldsymbol{x}) = \begin{pmatrix} \tilde{\psi}^1(\boldsymbol{x}) \\ \vdots \\ \tilde{\psi}^m(\boldsymbol{x}) \end{pmatrix}, \qquad (6)$$

where

$$\tilde{\psi}^j(\boldsymbol{x}) = \prod_{i=1}^{n} \psi_i^j(x^i), \qquad (7)$$

where ψ_i^j for each j are solutions of (systems) of ODEs in variables x^i.

Invariant characterization

A valence p symmetric tensor $K_{i_1,\ldots,i_p}$ that satisfies $\nabla_{(\mu_1}K_{\mu_2,\ldots,\mu_{p+1})}=0$ is called a Killing tensor. For $p=1$, K_μ is called a Killing vector.

Separability (in general non-orthogonal) of the HJ equation is characterized (Benenti 1997) by the existence of a pair (D_r, K), where D_r is an r-dimensional Abelian algebra of Killing vectors and K is a $p=2$ Killing tensor which satisfy the following conditions: (i) K is D_r-invariant with $m=n-r$ pointwwise and pairwise distinct real eigenvalues with orthogonally integrable eigenvectors. (ii) The submanifolds orthogonal to these eigenvectors are D_r-invariant. (r=0, yields the orthogonal case).

$\mathbb{K}$ is a symmetry operator of $\mathbb{H}$ iff $[\mathbb{K},\mathbb{H}]$=0.
The first and second order linear symmetry operators are defined by Killing vectors and valence two Killing tensors:
$\mathbb{K}:=K^\mu\nabla_\mu,\quad \mathbb{K}=\nabla_\mu K^{\mu\nu}\nabla_\nu$, where in the second case the following additional condition must be satisfied: $\nabla_\mu\left(K^{\rho[\mu}R^{\nu]}{}_\rho\right)=0$ where $R^\nu{}_\rho$ is the Ricci tensor.

The KG equation is separable iff the corresponding HJ equation is separable and the Robertson condition $K^{\rho[\mu}R^{\nu]}{}_\rho=0$ is satisfied. The separable solutions are eigenvectors of $\mathbb{K}$ with the separation constants as eigenvalues: $\mathbb{K}\psi=\lambda\psi$ (separation paradigm).

Invariant charcterization: Dirac

We would like to find an analogous characterization for Dirac of the following form:

Geometric conditions on M that imply there exists local coordinates x^i, a

spin transformation S,

which induces the following transformations on spinors and spin frames:

$$\psi' = S\psi \qquad e'^{\mu}_{a} = J^{\mu}_{\nu} e^{\nu}_{b} \ell^{b}_{a}$$

where ℓ is the image of S in $SO(\eta)$ and J is the Jacobian of the

transformation, and a separation matrix Y

such that the transformed Dirac equation, which reads

$$\mathbb{W}\psi' = 0,$$

where

$$\mathbb{W} = YS\mathbb{H}S^{-1},$$

is separable in the coordinates x^i.

Kerr solution

HJ and KG first separated in the stationary axisymmetric Kerr solution by Carter (1968) in Kerr-Newman coordinates. The separation is non-orthogonal and provides an illustration of Benenti's result. The separation also occurs in the algebraically simpler Boyer-Lindquist (BL) coordinates:

$$ds^2 = (1-\frac{2mr}{\Sigma})dt^2+\frac{4mar\sin^2\theta}{\Sigma}dtd\phi-\frac{(r^2+a^2)^2\sin^2\theta}{\Sigma}d\phi^2-\frac{\Sigma}{\Delta}dr^2-\Sigma d\theta^2,$$

where $\Sigma = r^2 + a^2\cos(\theta) \quad \Delta = r^2 - 2ma + a^2$. The metric possesses the commuting Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ (with corresponding momenta p_t and p_ϕ) and a $p = 2$, Killing tensor K defined by the quadratic first integral (see below) which underlies the separation property. One can show that $\{p_t, p_\phi\} = \{p_t, K\} = \{p_t, K\} = 0$, where $\{,\}$ denotes the Poisson bracket.

Carter shows that a solution of the form

$$S(t,\phi,r,\theta) = -Et + \Phi\phi + S_\theta + S_r,$$

separates the HJ equation and thus obtains a fourth constant of the motion for the geodesic equations. The solution to Hamilton's equations for the geodesics is thus obtained via Jacobi's theorem.

Kerr solution contd.

Dirac equation in the Weyl representation separated by Chandrasekhar (1976) WRT BL coordinates in the Kinnersley (null) spin frame.

Chandra procedure analysed by Carter & McLenaghan (1979) who employ BL coordinates and a symmetric (null) spin frame (Carter 1968).

$$\boldsymbol{\ell} = (2\Sigma\Delta)^{-1/2}((r^2+a^2)\partial_t+\Delta\partial_r+a\partial_\phi) \quad \boldsymbol{n} = (2\Sigma\Delta)^{-1/2}((r^2+a^2)\partial_t-\Delta$$

$$\boldsymbol{m} = (2\Sigma)^{-1}(-a\sin(\theta)\partial_t + i\partial_\theta - \csc(\theta)\partial_\phi),$$

They discovered the underlying commuting symmetry operators which characterize the separation and have the Chandra separation constants as eigenvalues:

$$\mathbb{K} = i(k^\mu - (1/4)\gamma^\mu\gamma^\nu\nabla_\nu k_\mu) \quad \mathbb{L} = i\gamma_5\gamma^\mu(f_\mu{}^\nu\nabla_\nu - (1/6)\gamma^\nu\gamma^\rho\nabla_\rho f_{\mu\nu})$$

where $\nabla_{(\mu}k_{\nu)} = 0$ and $f_{(\nu\rho)} = \nabla_{(\mu}f_{\nu)\rho} = 0$ (Killing-Yano equation).

The analysis also yields generalized formulae for the separated radial and angular parts (Carter & McLenaghan 1982) that include not only the full massive, charged Dirac equation but also the massive charged KG equation and the formulae for the neutral, zero mass spin $s = 1/2, 1, 2$ wave equations obtained by earlier workers .

Type D vacuum solution

- Most general first-order commuting symmetry operator (McLenaghan & Spindel 1979) is a sum of the two operators given previously and

$$\mathbb{M} = y^{\mu\nu\rho}\gamma_{\nu\rho}\nabla_{\mu} - (3/4)\nabla_{\mu}y^{\mu}\gamma_5$$

  where $y_{\mu\nu\rho} = y_{[\mu\nu\rho]}$ and $\nabla_{(\mu}y_{\nu)\rho\sigma} = 0$.

- Algebra of first symmetry operators investigated by McLenaghan & Spindel 1979.

- Chandra procedure extended to the class of Petrov type D vacuum solutions and its characterization by first order symmetry operators (Kamran & McLenaghan 1984).

- Most general first order R-commuting symmetry operators for the massless Dirac equation constructed (Kamran & McLenaghan 1984).

Factorizable systems

First general theory of Dirac separability is that of factorizable systems proposed by Miller (1988) as part of his general theory of mechanisms for variable separation in PDEs.
Miller studies systems of the form

$$\mathbb{D}\psi := H^i \partial_i \psi + V\psi = \lambda^1 \psi$$

A factorizable system for the above is set of n equations

$$\partial_i \psi = (C_{ij}\lambda^j - C_i)\psi$$

A factorizable system is separable if $\partial_k C_{ij} = \partial_k C_i = 0$ for $k \neq i$.

The integrability conditions for a factorizable are satisfied iff there exist a a system of $n-1$ first order differential operators that commute amongst themselves and with $\mathbb{D}$ and have the solutions of the fs as eigenspinors.

Non-factorizable systems

Fels & Kamran (1990) show

- that factorizable systems contain all the the separable systems for the Dirac equation on Petrov type D vacuum spacetimes.
- that for $n = 4$ there exist non-factorizable systems by giving an example of a metric with a 1-parameter isometry group and for which the Dirac equation admits a non-factorizable separable system characterized by second order symmetry operators. They also provide an example for $n = 2$.

It thus seems that a complete theory requires the study of second order symmetry operators.

The first steps in this direction were taken by McLenaghan, Smith & Walker (2000) and Smith (2002) who in the case $n = 4$ computed the general second-order symmetry operator using a two-component spinor formalism.

The remaining slides describe joint work with Carignano, Fatibene, Rastelli & Smith (2008-2015) for the case $n = 2$.

Separability in 2-dimensions: KG and H equations

If M admits a non-trivial valence two Killing tensor $\boldsymbol{K}$, there exists a coordinate system (u, v) such that

$$ds^2 = (A(u) + B(v)) \left(\mathrm{d}u^2 + \mathrm{d}v^2\right) \qquad \text{(Liouville metric)} \qquad (8)$$

$$\boldsymbol{K} = (A(u) + B(v))^{-1} \left(B(v)\partial_u \otimes \partial_u - A(v)\partial_v \otimes \partial_v\right)$$

where A and B are arbitrary smooth functions. The KG equation reads

$$\mathbb{H}\psi = (A + B)^{-1} \left(\partial^2_{uu}\psi + \partial^2_{vv}\psi\right) - E\psi = 0$$

Since $R_{\mu\nu} = 1/2Rg_{\mu\nu}$, $\mathbb{H}$ admits a second order symmetry operator $\mathbb{K}$:

$$\mathbb{K}\psi = (A + B)^{-1} \left(B\,\partial^2_{uu}\psi - A\,\partial^2_{vv}\psi\right)$$

With the product ansatz $\psi(u, v) = a(u)b(v)$, the KG equation, after multiplication by the separation factor $A + B$, reads $ba'' + ab'' - m(A + B)ab = 0$, which separates yielding $a''/a - mA = -b''/b + mB = \lambda$, where λ is a separation constant. It follows that separable solution satisfies

$$\mathbb{K}(ab) = \lambda ab$$

Dirac equation separability in 2-dimensions

A choice of gamma matrices valid for both signatures is

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \tag{9}$$

where $k = \sqrt{-\eta}$, $\eta = \det(\eta_{ab})$, and $\eta_{ab} = \mathrm{diag}(1, \pm 1)$ is the frame metric.

The Dirac equation may be written as

$$\mathbb{D}\psi := \tilde{A}\partial_1\psi + \tilde{B}\partial_2\psi + \tilde{C}\psi - \lambda\psi = 0 \tag{10}$$

where

$$\tilde{A} = \begin{pmatrix} A_1 & A_2 \\ -A_2 & -A_1 \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} B_1 & B_2 \\ -B_2 & -B_1 \end{pmatrix} \quad \tilde{C} = \begin{pmatrix} C_1 & -C_2 \\ C_2 & -C_1 \end{pmatrix}$$

with

$$A_1 = ie_1^1 \qquad A_2 = -ike_2^1 \qquad B_1 = ie_1^2 \quad B_2 = -ike_2^2 \tag{11}$$

$$C_1 = -(i/2)ke_2^\mu \Gamma_\mu^{12} \qquad C_2 = -(i/2)e_1^\mu \Gamma_\mu^{12} \tag{12}$$

Separation

With product separability assumption $\psi_i = a_i(x) b_i(y)$ the D equation reads

$$A_1 \dot{a}_1 b_1 + A_2 \dot{a}_2 b_2 + B_1 a_1 \dot{b}_1 + B_2 a_2 \dot{b}_2 + C_1 a_1 b_1 - C_2 a_2 b_2 - \lambda a_1 b_1 = 0$$

$$A_2 \dot{a}_1 b_1 + A_1 \dot{a}_2 b_2 + B_2 a_1 \dot{b}_1 + B_1 a_2 \dot{b}_2 - C_2 a_1 b_1 + C_1 a_2 b_2 + \lambda a_2 b_2 = 0$$

Definition

The Dirac equation is separable in the coordinates (x, y) if there exist a separation matix $diag(R_1(x,y), R_2(x,y))$ such the equations can be written as

$$R_1 a_i b_j (E_1^x + E_1^y) = 0 \quad R_2 a_k b_\ell (E_2^x + E_2^y) = 0$$

for a suitable choice of indices i, j, k, ℓ, where $E_i^x(x, a_j, \dot{a}_j)$ and $E_i^y(y, b_j, \dot{b}_j)$. Moreover, the equations

$$E_i^x(x, a_j, \dot{a}_j) = \mu_i = -E_i^y(y, b_j, \dot{b}_j)$$

define the separation constants μ_i.

Symmetry operators

We construct eigenvalue-type operators $\mathbb{L}\psi = \mu\psi$ with eigenvalues $\mu(\mu_i)$ making use only of the terms E_i^x and E_i^y. These operators are required to satisfy the following conditions:

- The operators $\mathbb{L}$ are independent of λ.
- $[\mathbb{L}, \mathbb{D}]\psi = 0$ for all ψ.
- $\lambda \neq 0$.

TYPES OF SEPARATION

I. $a_2 \neq a_2$ and $b_1 \neq b_2$

II. $a_1 = a_2 = a$ and $b_1 \neq b_2$ (or vive-versa)

III. $a_1 = a_2 = a$ and $b_1 = cb_2 = b$ (c constant)

Results

Proposition

The only type I separation is associated with the nonsingular Dirac operator and associated symmetry operator of the forms

$$\mathbb{D} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_x + (i/R_1(y)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_y \qquad \mathbb{L} = \begin{pmatrix} \partial_x^2 & 0 \\ 0 & \partial_x^2 \end{pmatrix}$$

The separable spin frame is given by

$$e_1^1 = e_2^2 = 0 \qquad e_1^2 = 1 \qquad e_2^1 = R_1(y)$$

The corresponding coordinates separate the geodesic Hamilton-Jacobi equation. If the Riemannian manifold is the Euclidean plane, the coordinates, up to a rescaling, coincide with polar or Cartesian coordinates.

Separability is also possible for equations of Type II which give rise to first order operators. Separability is not possible for Type III.

Conclusions from the analysis

- Dirac separability implies Hamilton-Jacobi Helmholtz separability.
- Dirac separability implies M admits a 1-parameter isometry group.

Second order symmetry operators

A second order symmetry operator for the Dirac equation is an operator of the form

$$\mathbb{K} = \mathbb{E}^{ab}\nabla_{ab} + \mathbb{F}^{a}\nabla_{a} + \mathbb{G}\mathbb{I},$$

which satisfies the defining relation

$$[\mathbb{K}, \mathbb{D}] = 0. \tag{13}$$

The coefficients $\mathbb{E}^{ab}, \mathbb{F}^{a}, \mathbb{G}$ are matrix zero-order operators and $\nabla_{ab} = \frac{1}{2}\left(\nabla_{a}\nabla_{b} + \nabla_{b}\nabla_{a}\right)$. The condition (13) is equivalent to

$$\begin{cases} \mathbb{E}^{(ab}\gamma^{c)} - \gamma^{(c}\mathbb{E}^{ab)} = 0 \\ \mathbb{F}^{(a}\gamma^{b)} - \gamma^{(b}\mathbb{F}^{a)} = \gamma^{c}\nabla_{c}\mathbb{E}^{ab} \\ \mathbb{G}\gamma^{a} - \gamma^{a}\mathbb{G} = \gamma^{c}\nabla_{c}\mathbb{F}^{a} - \frac{R}{4}\left(\mathbb{E}^{ab}\gamma^{c} + \gamma^{c}\mathbb{E}^{ab}\right)\epsilon_{bc}\gamma + \frac{R}{6}\left(\mathbb{E}^{bd}\gamma^{c} + 2\gamma^{c}\mathbb{E}^{bd}\right)\epsilon^{a}{}_{d}\epsilon \\ \gamma^{a}\nabla_{a}\mathbb{G} = \frac{R}{8}\left(\mathbb{F}^{a}\gamma^{b} + \gamma^{b}\mathbb{F}^{a}\right)\gamma\epsilon_{ab} + \frac{1}{12}\left(2\mathbb{E}^{ab}\gamma^{c} + \gamma^{c}\mathbb{E}^{ab}\right)\gamma\epsilon_{ac}\nabla_{b}R \end{cases} \tag{14}$$

where R denotes the Ricci scalar of $g_{\mu\nu}$.

Solution of defining equations

Expand $\mathbb{E}^{ab}$ in the basis ($\mathbb{I}$, γ_1, γ_2, γ) of the Clifford algebra C(2):

$$\mathbb{E}^{ab} = e^{ab}\mathbb{I} + e^{ab}_c\gamma^c + \hat{e}^{ab}\gamma,$$

where the coefficients e^{ab}, e^{ab}_c, $\hat{e}^{ab}$ are point functions in M. The solution of the first equation of (14) is given by

$$\mathbb{E}^{ab} = e^{ab}\mathbb{I} + 2\alpha^{(a}\gamma^{b)},$$

where the e^{ab} and the α^a are the frame components of arbitary valence two tensor and vector fields.
Setting

$$\mathbb{F}^a = f^a\mathbb{I} + f^a_b\gamma^b + \hat{f}^a\gamma,$$

we find that the most general solution of the first and second equations of (14) may be written as

$$\begin{cases} \mathbb{E}^{ab} = K^{ab}\mathbb{I} + 2\alpha^{(a}\gamma^{b)} \\ \mathbb{F}^a = f^a\mathbb{I} + (\gamma^c\nabla_c\alpha^a + A\gamma^a) + \frac{1}{3}\epsilon_{bc}\nabla^b K^{ac}\gamma \end{cases}$$

Solution of defining equations

where

$$\begin{cases} \nabla^{(a}K^{bc)} = 0 \\ \nabla^{(a}\alpha^{b)} = 0 \end{cases}$$

Observe that K^{ab} and is necessarily a valence two Killing tensor and α^a a Killing vector. Writing

$$\mathbb{G} = g\mathbb{I} + g_a\gamma^a + \hat{g}\gamma$$

we find that the most general solution of the first three equations of (14) is given by

$$\begin{cases} \mathbb{E}^{ab} = K^{ab}\mathbb{I} + 2\alpha^{(a}\gamma^{b)} \\ \mathbb{F}^a = (\zeta^a + \nabla_c K^{ac})\mathbb{I} + (\gamma^c\nabla_c\alpha^a + A\gamma^a) + \frac{1}{3}\epsilon_{bc}\nabla^b K^{ac}\gamma \\ \mathbb{G} = g\mathbb{I} - \frac{R}{4}\alpha_b\gamma^b + \frac{1}{4}\epsilon_{ba}\nabla^b\zeta^a\gamma \end{cases}$$

where

$$\begin{cases} A \in \mathbb{C} \\ \nabla^{(d}K^{ab)} = 0 \\ \nabla^{(a}\alpha^{b)} = 0,\ \nabla^{(d}\zeta^{b)} = 0 \end{cases}$$

Solution of defining equations

Finally the fourth equation of (14) yields

$$\nabla_a g = -\frac{1}{4}\nabla_b \left(R K_a^{\cdot b}\right) \quad \Rightarrow \partial_\mu g = -\frac{1}{4}\nabla^\nu \left(R K_{\mu\nu}\right) \tag{15}$$

This equation locally detemines g and hence the most general second order symmetry operator of the Dirac equation if and only if the right hand side is a closed 1-form. If the space is flat, $R = 0$ and the solution of is $g = constant$. The integrability condition for (15) is

$$\nabla_{[\mu}\nabla^\rho \left(R K_{\nu]\rho}\right) = 0 \tag{16}$$

CONCLUSION:

The most general second order symmetry operator is defined by a Killing tensor $K_{\mu\nu}$, two Killing vectors α_μ, ζ_μ, a scalar function g, and a constant A, provided these objects exist on M.

Integrability condition solution

For the Liouville metric (8) The integrability condition (16) reads

$$(A+B)^2(A'B'''+A'''B')+6A'B'\left((A')^2+(B')^2\right)-6A'B'(A+B)(A''+B'')=0 \tag{17}$$

If one of A or B is constant, (17) is trivially satisfied in which case the space admits a Killing vector. In the case $A'B' \neq 0$, the integrability condition (17) implies that

$$\begin{cases} (A')^2 = kA^4 + a_3A^3 + a_2A^2 + a_1A + a_0 \\ (B')^2 = -kB^4 + a_3B^3 - a_2B^2 + a_1B - a_0 \end{cases}$$

where a_i, $i = 0, 1, 2, 3$, and k are arbirary constants. If A and $-B$ are taken as coordinates, the metric may be written as

$$ds^2 = (A-B)\left(\frac{dA^2}{p_4(A)} - \frac{dB^2}{p_4(B)}\right), \tag{18}$$

where $p_4(A) := kA^4 + a_3A^3 + a_2A^2 + a_1A + a_0$.
The Ricci scalar is given by $R = (A+B)k + \frac{1}{2}a_3$. If $k = 0$, (18) is the metric of a space of constant curvature.

First order operators and reducibility

A first order operator is said to be trivial if it has the form $\alpha\mathbb{D}+\beta\mathbb{I}$.
Thus the most general first order operator has the form

$$\begin{cases} \mathbb{E}^{ab}=0 \\ \mathbb{F}^{a}=\zeta^{a}\mathbb{I}+A\gamma^{a} \\ \mathbb{G}=g\mathbb{I}+\frac{1}{4}\epsilon_{ba}\nabla^{b}\zeta^{a}\gamma \end{cases} \qquad \text{where} \begin{cases} A,g\in\mathbb{C} \\ \nabla^{(d}\zeta^{b)}=0 \end{cases}$$

There is a one-to-one correspondence between non-trivial first order operators and Killing vectors ζ on M. A second order operator is said to be trivial if it has the form $\mathbb{D}\circ\mathbb{K}_1+\mathbb{K}_1'$, where $\mathbb{K}_1$ and $\mathbb{K}_1'$ denote any first order operators.
Thus the most general non-trivial second order operator has the form

$$\begin{cases} \mathbb{E}^{ab}=K^{ab}\mathbb{I} \\ \mathbb{F}^{a}=\nabla_{c}K^{ac}\mathbb{I}+\frac{1}{3}\epsilon_{bc}\nabla^{b}K^{ac}\gamma \\ \mathbb{G}=g\mathbb{I} \end{cases} \qquad \text{where} \begin{cases} \nabla^{(d}K^{ab)}=0 \\ K^{ab}\neq\lambda\eta^{ab} \\ \nabla_{[\mu}\left(\nabla^{\lambda}\left(RK_{\nu]\lambda}\right)\right)=0 \end{cases}$$

and where g a solution of equation (16). There is a one-to-one correspondence between non-trivial second order operators and Killing tensors K^{ab} on M such that $K^{ab}\neq\lambda\eta^{ab}$ and $\nabla_{[\mu}\left(\nabla^{\lambda}\left(RK_{\nu]\lambda}\right)\right)=0$.

Separation of variables

It is known that the Dirac equation separates only in coordinate systems separating the Helmholtz equation for the corresponding metric tensor, and only if one at least of these coordinates is associated to a Killing vector.

This is the case for the coordinate system (u, v) for which the metric is in Liouville form, and u is an ignorable coordinate. The appropriate spin-frame in this case is given by

$$(e_a^\mu) = \begin{pmatrix} 0 & \frac{1}{\sqrt{A(u)+B(v)}} \\ -\frac{1}{\sqrt{A(u)+B(v)}} & 0 \end{pmatrix},$$

where $A = 0$ and $B(v) = \beta(v)^{-2}$. Assuming that vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\zeta}$ are zero the Dirac operator $\mathbb{D}$ and second order symmetry operator $\mathbb{K}$ may be written in matrix form as

$$\mathbb{D} = \beta \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_u + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \partial_v \right] - \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \beta',$$

and

$$\mathbb{K} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial^2_{uu}.$$

Separation of variables

The expressions for $\mathbb{D}$ and $\mathbb{K}$ coincide with the separation scheme earlier. It is an instance of non-factorizable separation of Fels & Kamran. By computing

$$\mathbb{D}\psi - m\psi = 0$$

and

$$\mathbb{K}\psi - \mu\psi = 0 \tag{19}$$

with the separation ansatz

$$\psi = \begin{pmatrix} a_1(u)b_1(v) \\ a_2(u)b_2(v) \end{pmatrix}$$

we obtain the separated equations

$$\begin{cases} -\frac{a_2'}{a_1} = -i\frac{b_1'}{b_2} + \frac{i\beta'}{2\beta}\frac{b_1}{b_2} + \frac{m}{\beta}\frac{b_1}{b_2} \\ \frac{a_1'}{a_2} = i\frac{b_2'}{b_1} - \frac{i\beta'}{2\beta}\frac{b_2}{b_1} + \frac{m}{\beta}\frac{b_2}{b_1} \end{cases}$$

and, from (19) we have

$$\begin{cases} a_1'' = -\mu a_1 \\ a_2'' = -\mu a_2 \end{cases} \tag{20}$$

Separation of variables

By introducing separation constants μ_1 and μ_2 such that $\mu_1\mu_2 = \mu$, the first system gives for the a_i

$$\begin{cases} a_2' = -\mu_1 a_1 \\ a_1' = \mu_2 a_2 \end{cases} \tag{21}$$

and for the b_i

$$\begin{cases} -ib_1' + i\frac{\beta'}{2\beta} b_1 + \frac{m}{\beta} b_1 = \mu_1 b_2 \\ ib_2' - i\frac{\beta'}{2\beta} b_2 + \frac{m}{\beta} b_2 = \mu_2 b_1 \end{cases} \tag{22}$$

It is evident that $\mathbb{K}$ provides, via (20), a decoupling relation for the equations (21). On the other hand, (20) can be obtained by applying twice equations (21). The symmetry operators associated with separation are generated by this means in MR. The solutions for the a_i are

$$\begin{cases} a_1 = c_1 \sin(\sqrt{\mu} u) + c_2 \cos(\sqrt{\mu} u) \\ a_2 = \sqrt{\frac{\mu_1}{\mu_2}} [-c_2 \sin(\sqrt{\mu} u) + c_1 \cos(\sqrt{\mu} u)] \end{cases}$$

where $c_1, c_2 \in \mathbb{C}$.

Separation of variables

The general solution of (22) can be easily computed in Cartesian coordinates, $\beta(v) = 1$:

$$\begin{cases} b_1 = d_1 \sin(Mv) + d_2 \cos(Mv) \\ b_2 = \frac{1}{\mu_1}[(d_1 + iMd_2)\sin(Mv) + (d_2 - iMd_1)\cos(Mv)] \end{cases}$$

where $d_1, d_2 \in \mathbb{C}$ and $M = \sqrt{m^2 - \mu}$. It is remarkable that, even if the a_i and the b_i depend on the μ_j, the products $\psi_i = a_i b_i$ depend on μ only.

By setting β equal to 1, e^v, $\sinh(v)$, $\cosh(v)$, $k - \cos(v)$, with $k > 1$, respectively, the corresponding Riemannian manifolds are:

- the Euclidean plane in Cartesian and polar coordinates
- the sphere
- the pseudo-sphere
- the torus

For polar coordinates in the Euclidean plane and in the cases of sphere and pseudosphere, equations (22) can be integrated, to obtain solutions respectively in terms of Bessel functions and, on the sphere and pseudosphere, in terms of hypergeometric functions.

Future directions

- Give a complete invariant characterization for Dirac equation separability comparable to that for the Hamilton-Jacobi and Helmholtz equations.

- Given that the invariant conditions for separability are satisfied, determine a procedure for the transformation to separable form.

- Extend the theory to higher dimensions.

References

- B. Carter, Phys. Rev. 174, 1559, 1968.
- B. Carter, Commun. Math. Phys. 10, 280, 1968.
- B. Carter, J. Math. Phys. 10, 70, 1969.
- B. Carter, Phys. Rev. D 16, 3395, 1977.
- S. Chandrasekhar, Proc. Roy. Soc. London A349, 573, 1976.
- B. Carter & R.G. McLenaghan, Phys. Rev. D 19, 1093, 1979; in Proceedings of the 2nd Marcel Grossmann Meeting on GR, Part A, North-Holland, 552, 1982.
- R.G. McLenaghan & Ph. Spindel, Phys. Rev. D 20, 409, 1979; Bull. Soc. Math. Belg. XXXI, 65, 1979.
- N. Kamran & R.G. Mclenaghan, J. Math. Phys. 25, 1019, 1984; Phys. Rev. D, 357, 1984; Bull. Cl. Sci. Acad. R. Belg. LXX, 596, 1984.
- W. Miller, Jr., In Symmetries and Nonlinear Phenomena, World Scientific, Singapore, 188, 1988.
- N. Kamran, M. Legaré, R. G. McLenaghan & P. Winternitz, J. Math. Phys. 29, 1988.
- M. Fels & N. Kamran, Proc. Roy. Soc. London A 428, 229, 1990.
- S. Benenti, J. Math. Phys. 38, 6578, 1997.

References contd.

- R.G. McLenaghan, S.N. Smith, & D.M. Walker, Proc. Roy. Soc. London A 456, 2629, 2000.
- R.G. McLenaghan & G. Rastelli, In IMA volumes in mathematics and its applications, Vol. 144, 471, Springer 2008
- L. Fatibene, R.G. McLenaghan, G. Rastelli, & S. N. Smith, J. Math. Phys. 50, 053516, 2009.
- A. Carignano, L. Fatibene, R.G. McLenaghan & G. Rastelli, SIGMA 7 (057), 2011.
- L. Fatibene, R.G. McLenaghan & G. Rastelli, Int. J. Math. M. Mod. Phys. 12, 1550012-1, 2015.