Bayesian Reasoning
From Carter Catastrophe to Testing No-Hair Theorem

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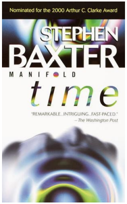
Mini Bio

–1996:	Study physics in Graz (Austria)
1996–1997:	Erasmus program, Paris
	DEA physique théorique @ École Normale
1997–2000:	PhD @ Meudon: Brandon & D. Langlois
	☞ superfluid/multi-fluid hydrodynamics
	("Aspects de l'hydrodynamique superfluide des étoiles à neutrons")
2000–2003:	Southampton: N. Andersson
	☞ superfluid neutron-star models & oscillation modes
2003–2007:	Albert-Einstein-Insitute Golm: C. Cutler, M.A. Papa
2007–	AEI Hannover: M.A. Papa, B. Allen
	☞ data analysis of LIGO/Virgo data: searches for *continuous-gravitational-waves* from neutron stars
	☞ Bayesian probability

Carter catastrophe?



'We don't expect to find ourselves in a special place in *space*. Why should we expect to be in a special place in *time*? But that is what you have to accept, you see, if you believe mankind has a future with very distant limits. Because in that case we must be among the very first humans who ever lived …'

'Get to the point,' Malenfant said softly.

'… All right. Based on arguments like this, we think a catastrophe is awaiting mankind. A universal extinction, a little way ahead.

'We call this the Carter catastrophe.'

thirty years of these studies behind us now. The methodology was first proposed by a physicist called Brandon Carter in a lecture to the Royal Society in London in the 1980s. And we have built up estimates based on a range of approaches,

[B. Carter, "The anthropic principle and its implications for biological evolution" Philos. Trans. R. Soc. A (1983)]

Argument developed in: [J. Leslie, (1990a, 1990b, 1992a, 1992b, 1993]

The box argument analogy

Cornelius reached under the table and produced a wooden box, sealed up. It had a single grooved outlet, with a wooden lever alongside. ‘In this box there are a number of balls. One of them has your name on it, Malenfant; the rest are blank. If you press the lever you will retrieve the balls one at a time, and you may inspect them. The retrieval will be truly random.

‘I won’t tell you how many balls the box contains. I won’t give you the opportunity to inspect the box, save to draw out the balls with the lever. But I promise you there are either ten balls in here – or a thousand. Now. Would you hazard which is the true number, ten or a thousand?’

‘Nope. Not without evidence.’

‘Very wise. Please, pull the lever.’

Malenfant drummed his fingers on the table top. Then he pressed the lever.

A small black marble popped into the slot. Malenfant inspected it; it was blank. Emma could see there was easily room for a thousand such balls in the box, if need be.

Malenfant scowled and pressed the lever again.

His name was on the third ball he produced.

“Box” = humanity, “you” being drawn ☞ doom imminent?

[Bernoulli, Bayes, Laplace]

> Probability theory is nothing but common sense reduced to calculation.
>
> *Laplace, 1819*

[Jaynes "Probability theory. The Logic of Science"(2003)]

Probability: *extension* of deductive logic to *incomplete information*

A, I: logical propositions $\in \{\text{True}, \text{False}\}$

$P(A|I) \equiv$ *plausibility* of A being True *given/assuming* I is True

$P(A|I)$ quantifies observer's knowledge about statement A

- ☞ not: random variables, limiting frequencies, sets
- ☞ not a property of the observed system! [Jaynes: "Mind projection fallacy"]

The Three Laws of Probability

[Cox (1946, 1961), Jaynes] Requiring three *desiderata* for $P(A|I)$:
(i) $P \in \mathbb{R}$, (ii) consistency, (iii) agreement with "common sense"
one can *derive* unique laws of probability:

1. $P(A|I) \in [0,1]$ $\quad \begin{cases} P(A|I) = 1 & \Leftrightarrow \text{certain } (A|I) \text{ is True} \\ P(A|I) = 0 & \Leftrightarrow \text{certain } (A|I) \text{ is False} \end{cases}$
2. $P(A|I) + P(\neg A|I) = 1$
3. $P(A \text{ and } B|I) = P(A|B,I)\ P(B|I)$

Immediate consequences

- Bayes theorem: $P(A \text{ and } B|I) = P(B \text{ and } A|I)$

  ☞ $P(A|B,I) = P(B|A,I)\ \frac{P(A|I)}{P(B|I)}$
- $P(A \text{ or } B|I) = P(A|I) + P(B|I) - P(A \text{ and } B|I)$

  ☞ if exactly one of $\{A_i\}_{i=1}^{N}$ is true: $\sum_i P(A_i|I) = 1$

Model assumptions: (S. Baxter, J. Leslie)

$$I \equiv \begin{cases} I_1 : \text{opaque box containing } N \text{ balls} \\ I_2 : N \text{ is either } N = 10 \ \texttt{or} \ N = 1000, \text{ equally likely} \\ I_3 : \text{exactly one ball is "special" } (\mathbb{S}), \text{ at unknown position} \end{cases}$$

observation: $\mathbb{S}_k \equiv$ draw "special" ball on kth draw, ($k \leq 10$)

likelihood: $P(\mathbb{S}_k|N, I) = \frac{[N \geq k]}{N} = \begin{cases} \frac{1}{N} & \text{if } N \geq k \\ 0 & \text{otherwise} \end{cases}$

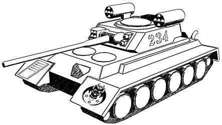
posterior: $P(N|\mathbb{S}_k, I) \propto P(\mathbb{S}_k|N, I)\ P(N|I) = \frac{[N \geq k]}{N}\, p(N)$

☞ posterior *odds*: $O \equiv \frac{P(N=10|\mathbb{S}_k, I)}{P(N=1000|\mathbb{S}_k, I)} = \frac{1000}{10} = 100$

☞ $P(N = 10|\mathbb{S}_k, I) = \frac{100}{101} \approx 99\,\%$

Box argument: applications

1. Observe a 'random' tank with a sequential serial number k:



How many tanks N?

☞ $P\left(N|\mathrm{S}_k, I\right) \propto \frac{[N \geq k]}{N}\, p(N)$

What if the tank has no (meaningful) serial number?

2. Doomsday near?

I_3: exactly one ball is "special" $\Leftrightarrow$ balls are numbered

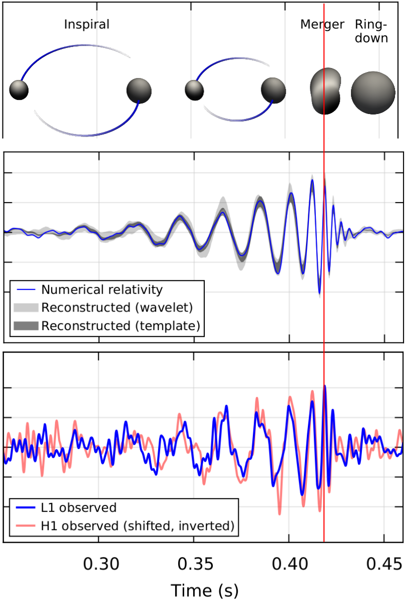
☞ Strong prior assumption! . . . what happens without I_3?

observation: B_k: k balls have been drawn

likelihood: $P\left(\mathrm{B}_k|N, I\right) = [N \geq k]$

☞ posterior $P\left(N|\mathrm{B}_k, I\right) \propto p(N)\ [N \geq k]$

Binary Black-Hole Merger GW150914



Best-matching GR waveform:

☞ final mass $M \sim 68\,\mathrm{M}_\odot$

dimensionless spin $a \sim 0.674$

merger-time

$t_M \sim 0.416\,\mathrm{s}@L1 \sim 0.423\,\mathrm{s}@H1$

But do we actually see a “ringdown” and is it consistent with inspiral+merger?

[LVC, “Observation of Gravitational Waves from a Binary Black Hole Merger”, PRL116 (2016)]

GW150914: Ringdown?

I_1: Measured strain: $x(t) = n(t) + s(t)$, $\quad n \sim$ Gaussian noise

I_2: Ringdown waveform (let $t = 0$ at $t_M + \Delta t$)

$$s(t; \mathcal{A}, f, \tau) = A\, e^{-\frac{t}{\tau}} \cos(2\pi f\, t + \phi_0), \qquad \mathcal{A} \equiv \{A, \phi_0\}$$

☞ likelihood $P\,(x|\mathcal{A}, f, \tau, I) \propto e^{-\frac{1}{2}(x-s|x-s)}$

1. Test *signal* hypothesis ($A > 0$) versus noise ($A = 0$):
   - ☞ Bayes factor: $B_{S/G} = \frac{P(x|A>0,I)}{P(x|A=0,I)} = \ldots$
2. Estimate f and τ (where $\mathcal{A}$ are “nuisance” parameters)
   - ☞ posterior: $P\,(f, \tau|x, I) = \int P\,(f, \tau, \mathcal{A}|x, I)\; d\mathcal{A}$

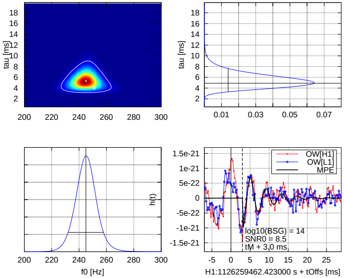
$$\propto \int P\,(x|\mathcal{A}, f, \tau, I)\, P\,(\mathcal{A}, f, \tau|I)\, d\mathcal{A}$$

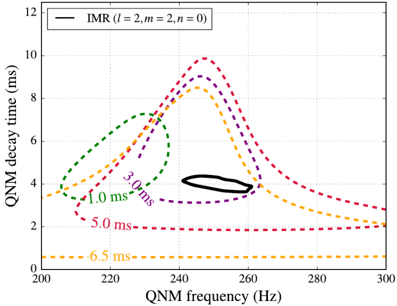
GW150914: The sound of one black hole ringing

- test of *No-hair theorem*: only BH parameters M, a

[B. Carter, "Axisymmetric Black Hole Has Only Two Degrees of Freedom", PRL26 (1971)]

- GR predicts dominant (220) mode: $f \sim 250\,\mathrm{Hz}$, $\tau \sim 4\,\mathrm{ms}$
  QNM $\Delta t \sim 10 - 20\,\mathrm{M}_\odot \sim 3 - 7\,\mathrm{ms}$ after merger t_M

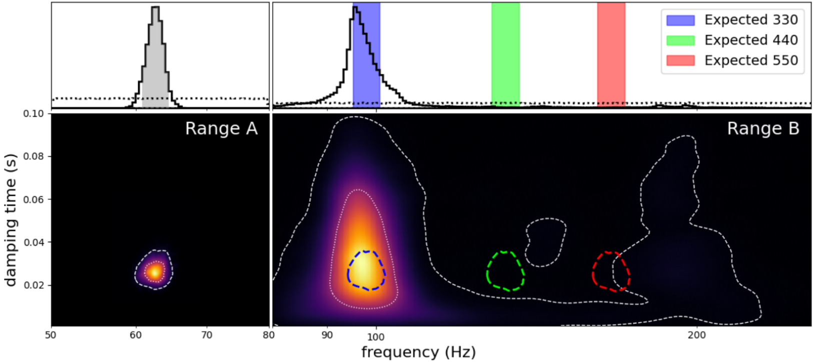




[LVC, "Tests of General Relativity with GW150914", PRL116 (2016)]

GW190521: Black-hole spectroscopy

Binary black-hole merger: final $M \sim 330\,\mathrm{M}_\odot$, $a \sim 0.87$

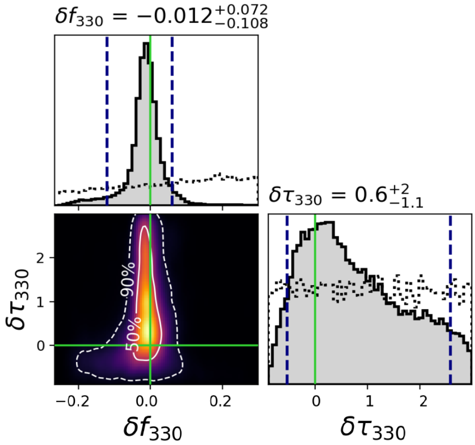


[Capano et al., "Observation of a multimode quasi-normal spectrum from a perturbed black hole", arXiv:2105.05238 (2021)]

Quantitative Test of No-Hair Theorem

$\{f_{220}, \tau_{220}\} \Rightarrow \{M, a\} \Rightarrow \{f_{330}, \tau_{330}\}$

☞ allow deviations from GR: $f_{330}\,(1 + \delta f_{330})$, $\tau_{330}\,(1 + \delta\tau_{330})$



[Capano et al., "Observation of a multimode quasi-normal spectrum from a perturbed black hole", arXiv:2105.05238 (2021)]

Thank you

Thank you, and
Happy Birthday Brandon!