A visual journey into some Carter-Penrose diagrams

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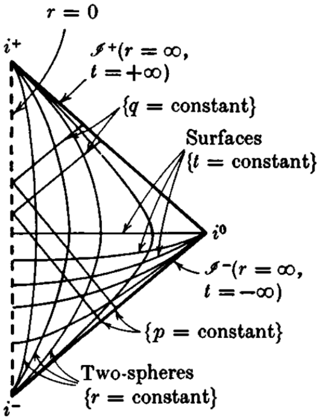
Introduction

So...

- Happy birthday Brandon !
- Fasten your seat belt
- Enjoy the trips



$r = 0$
i^+
$\mathscr{I}^+(r = \infty,$
$t = +\infty)$
$\{q = \text{constant}\}$
Surfaces
$\{t = \text{constant}\}$
i^0
$\mathscr{I}^-(r = \infty,$
$t = -\infty)$
$\{p = \text{constant}\}$
Two-spheres
$\{r = \text{constant}\}$
i^-

Minkowski has a rather simple causal structure.

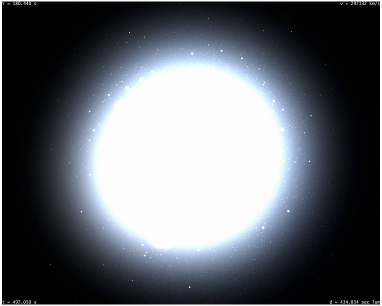
Its aspect does not depend on observer's position, but does depend on observer's velocity

Changing velocity will induce :

- Doppler shift
- Intensity shift
- Position shift (aberration)

Some example : lets us accelerate from 0 to 0.995 c in 200 seconds

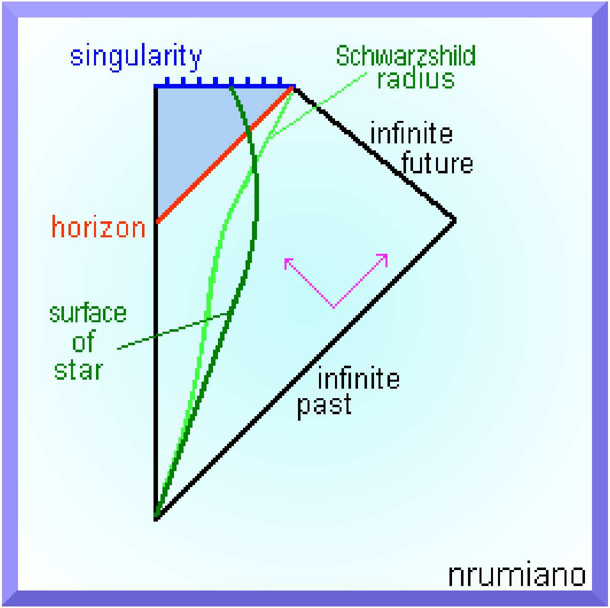
Warning ! $a \sim 450\ 000\ g_{\mathrm{E}}$ (To compare with big formula 1/Indycar crash : 300 km/h to 0 in 2 meters $\rightarrow a \sim 214\ g_{\mathrm{E}}$ for 40 ms (Kenny Bräck, Texas Motor Speedway, 2003)



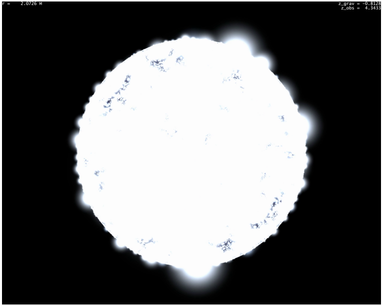
A few things to remember :

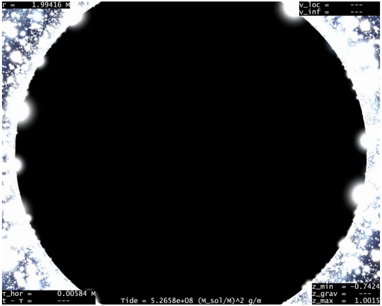
- fast space travel is boring : either you don't see anything, or you are blinded by intense flux
- Doppler and intensity shifts are a nuisance. They will be attenued, sometimes completely removed in what follows

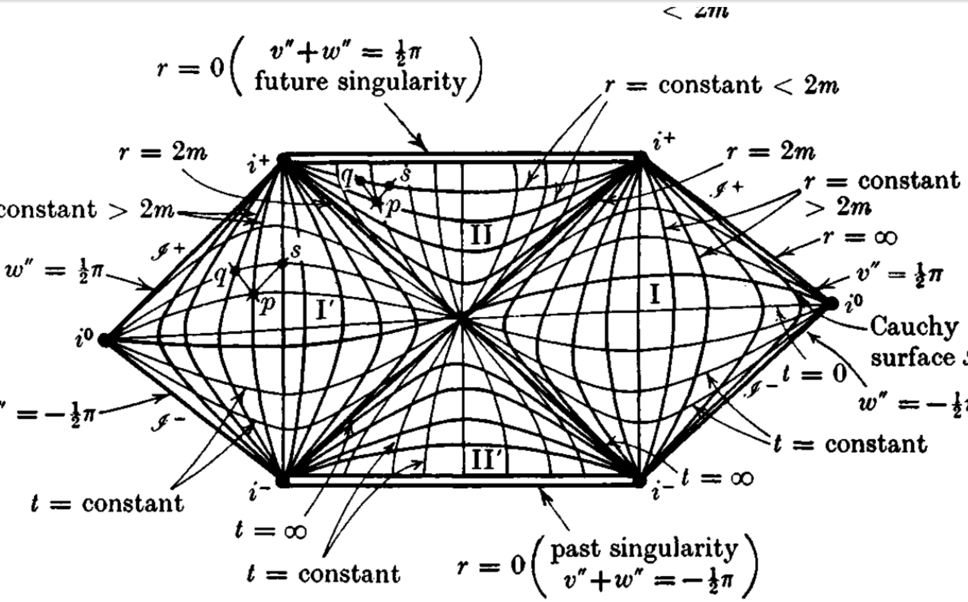
Part 2 : Schwarzschild, astrophysical



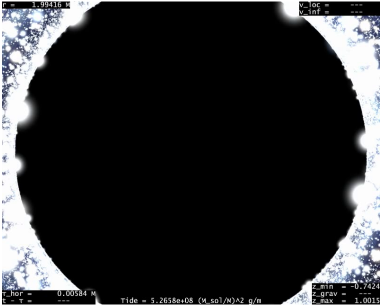
- Adding a black hole in the picture causes gravitational distorsions, which shall add to the special relativistic (Minkowskian) effects.
- These effects are increasing confusing as you get closer and closer to the black hole
- In particular it is hard to tell by eye whether you are inside or outside thz black hole (because of aberration)

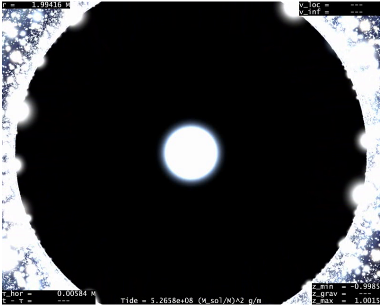





< 2m
r = 0 (v″ + w″ = ½π future singularity)
r = constant < 2m
r = 2m
i+
q
s
p
II
i+
r = 2m
r = constant > 2m
constant > 2m
𝒥+
r = constant > 2m
r = ∞
w″ = ½π
s
q
p
I′
I
v″ = ½π
i0
i0
Cauchy surface
t = 0
w″ = −½π
𝒥−
w″ = −½π
𝒥−
t = constant
II′
i−
i−
t = ∞
t = constant
t = ∞
t = constant
r = 0 (past singularity v″ + w″ = −½π)

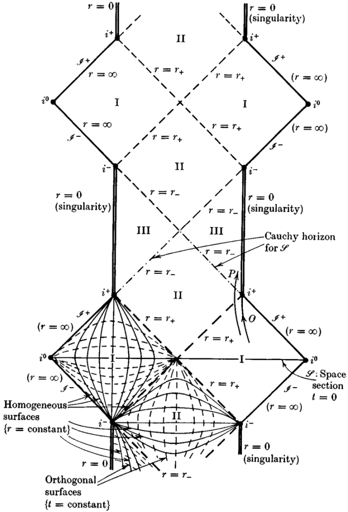
- The whole analytic extension of the metric is more symmetric than its astrophysical counterpart.
- Contrarily to the common claim, horizon crossing is accompanied with observable effets.





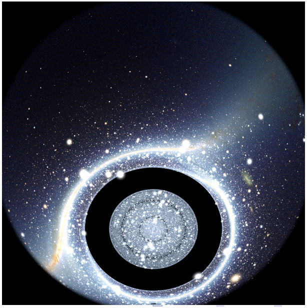
- The non static nature of black hole interor becomes more obvious

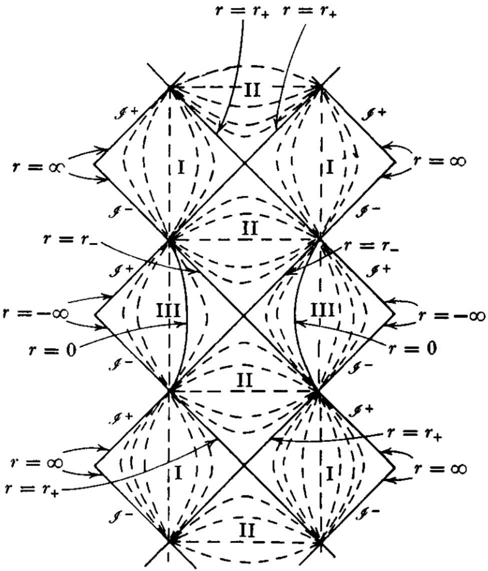
Part 4 : Reissner-Nordtröm



Many new features arise in the RN case :

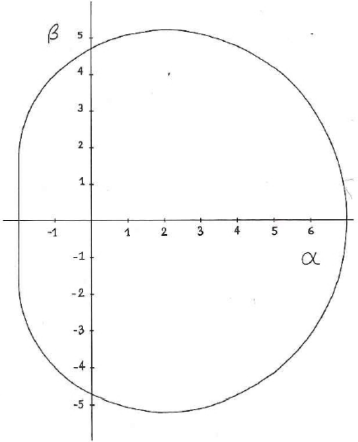
- Singularity is timelike. It is a point that cannot be reached by almost any geodesic : il is invisible !
- There is not endpoint to any geodesic : if you cross the horizon, you'll cross it back and exit somewhere else. This is no longer a black hole, but a wormhole (or a naked singularity).
- There are some bounded null geodesics. If you assume that photons arise only from past null infinities, then you may cross dark regions.



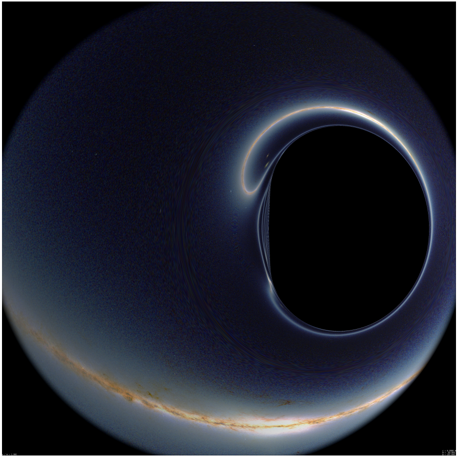


$r = r_+$ $r = r_+$
II
$\mathscr{I}^+$
$r = \infty$
I
$\mathscr{I}^-$
$r = r_-$
$r = -\infty$
III
$r = 0$
$r = r_+$

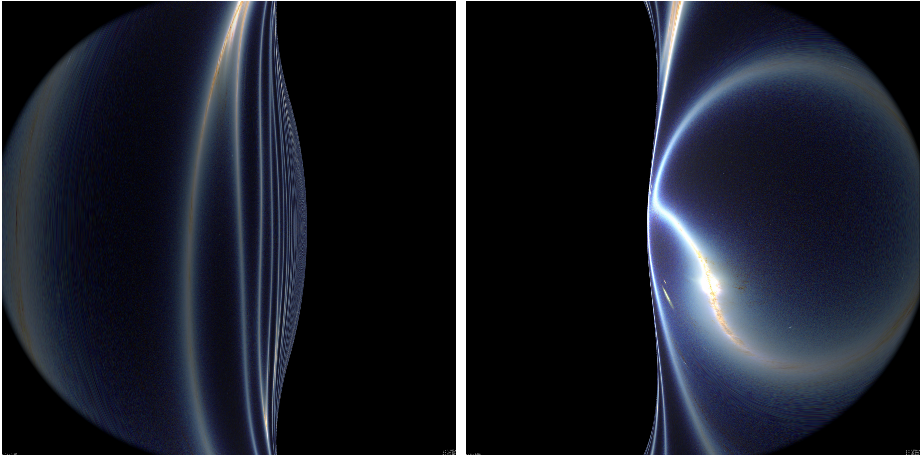
- Even the exterior of the Kerr metric has some hidden surprises.
- We all know of the elongeted, asymmetric shape of an extremal Kerr BH seen from the equatorial plane and from a large distance :



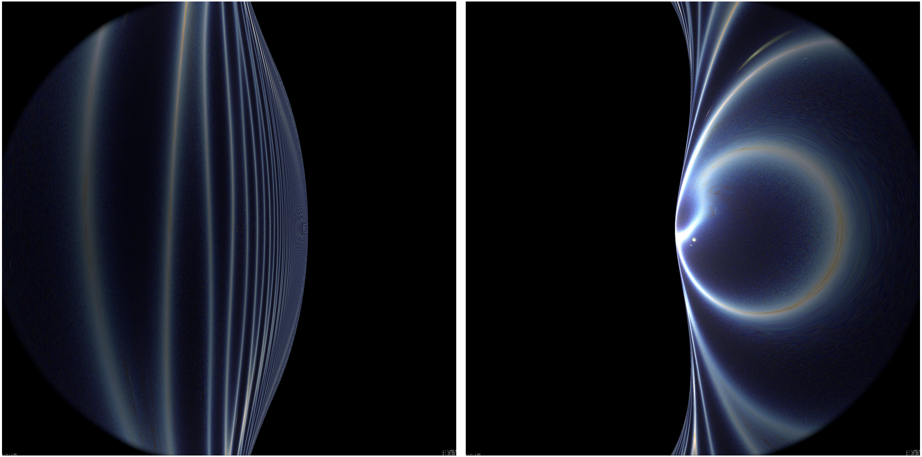
- But what happens if you look at it from close distance ?



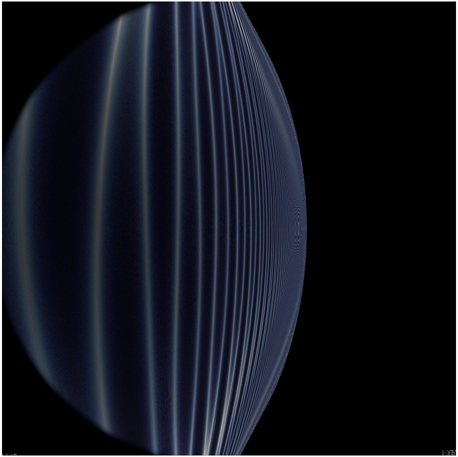
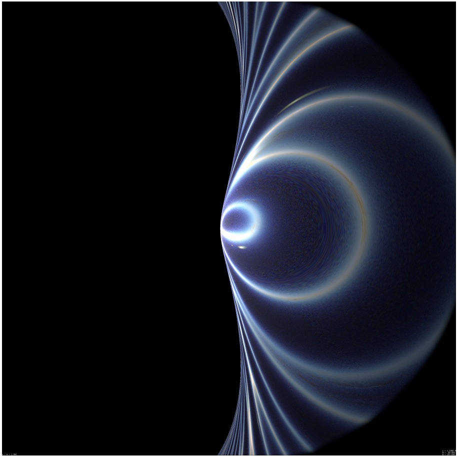
$r = 6M$



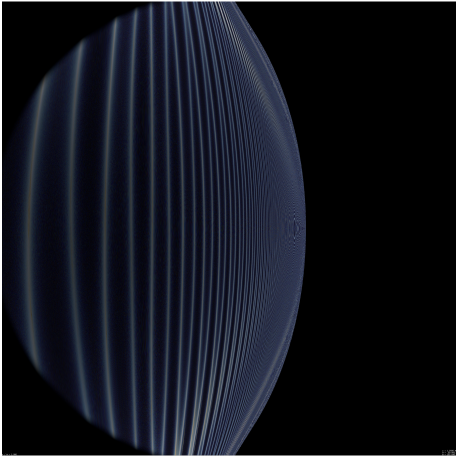
$r = 1.5M$

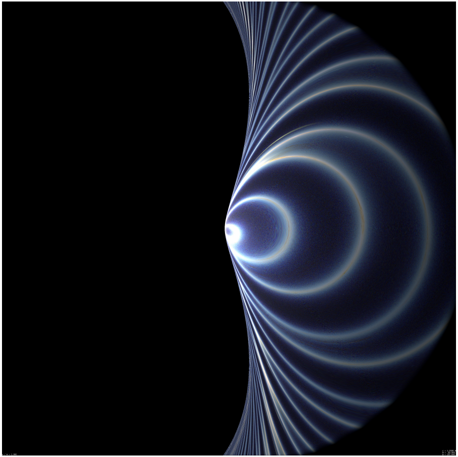


$r = 1.2M$

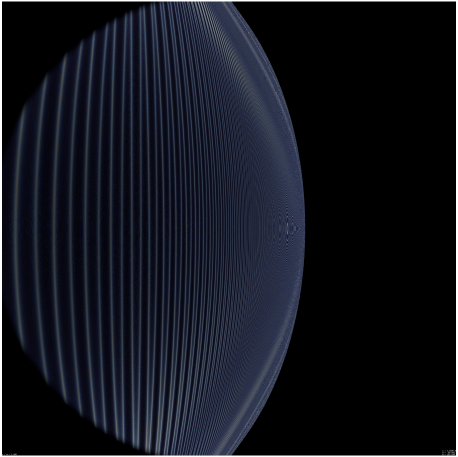
 

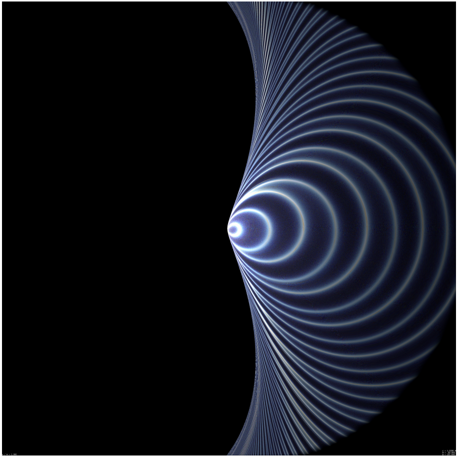
$r = 1.1M$



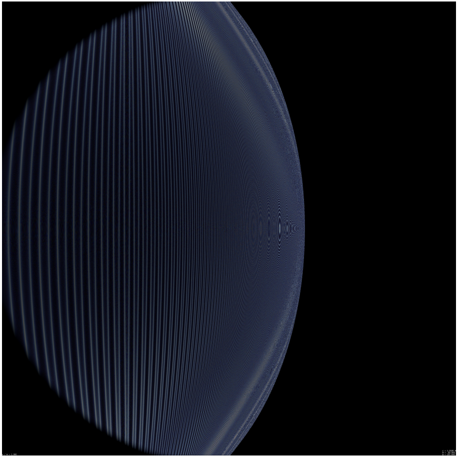
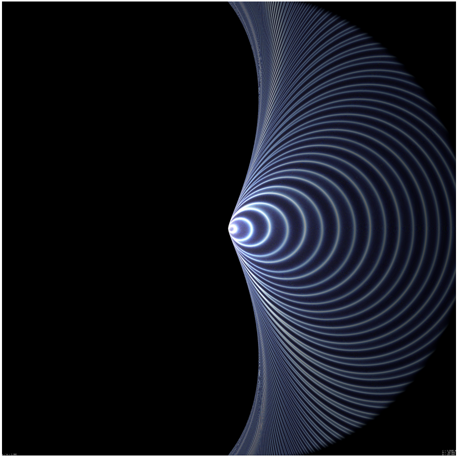


$r = 1.05M$

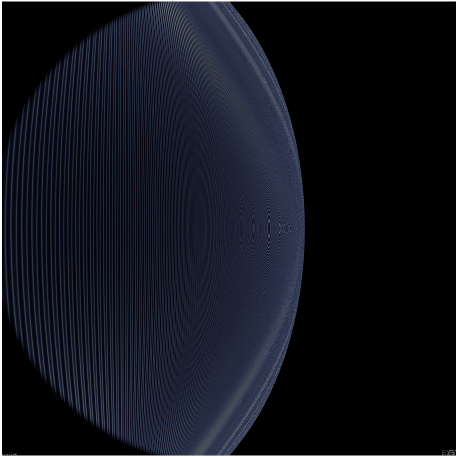
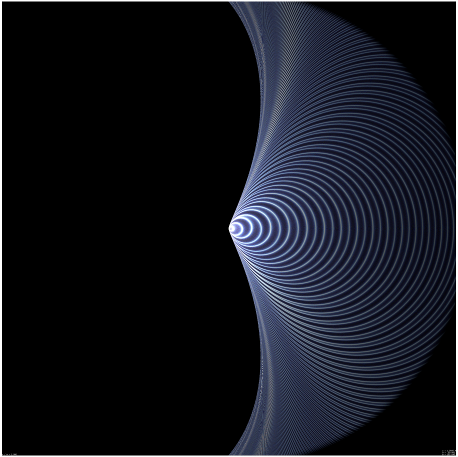




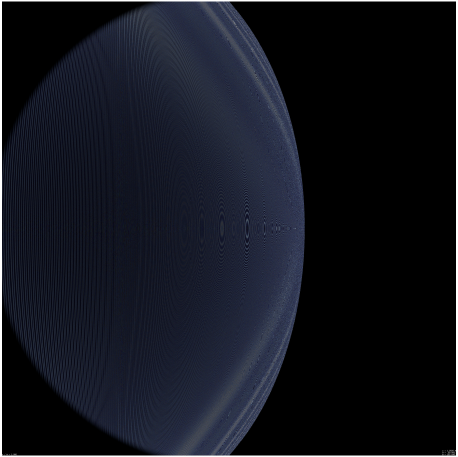
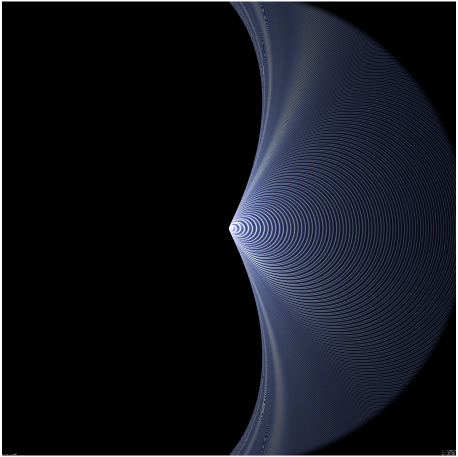
$r = 1.02M$

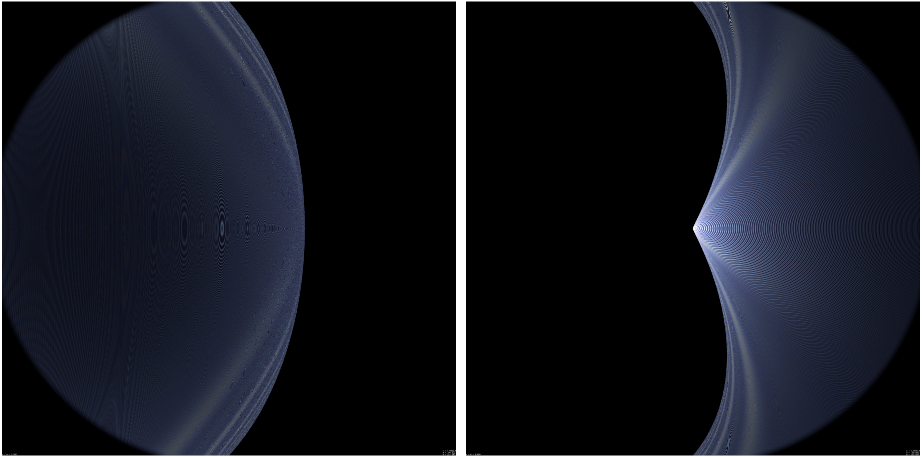
$r = 1.01M$

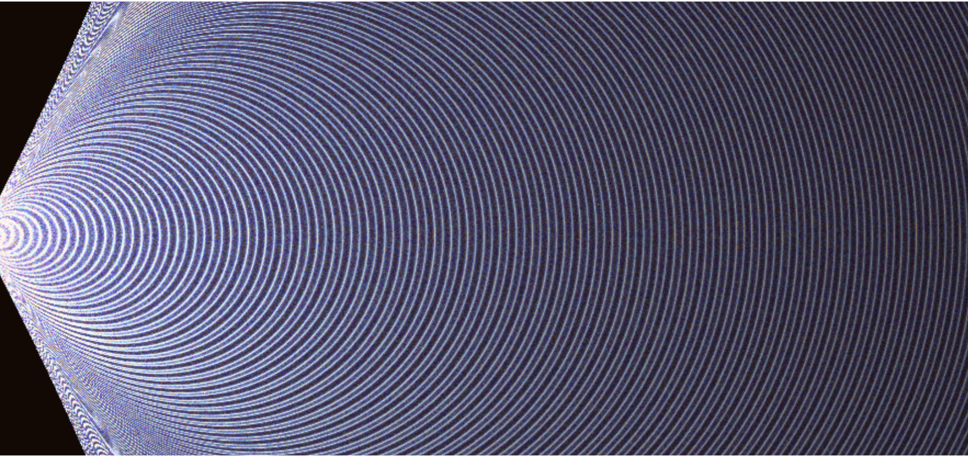
$$r = 1.005M$$

$r = 1.002M$



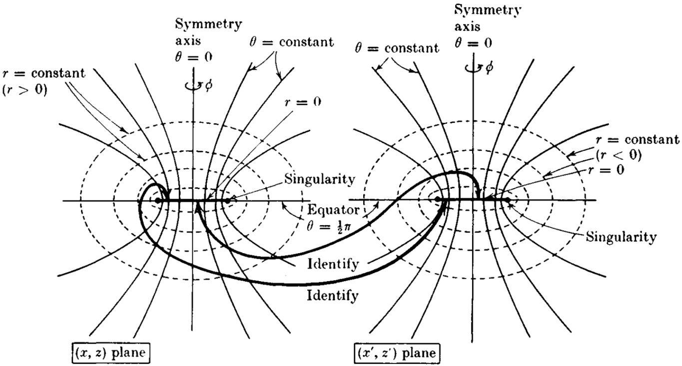
$r = 1.001M$



$r = 1.0005M$, zoomed 50° field of view

This was funny, but it was a continuation of Bardeen's work, not yours. Yours dealt (among other things) with causal structure and ring singularity.

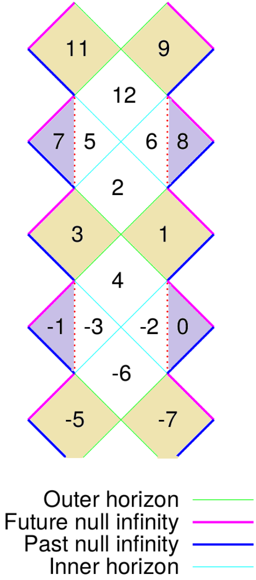
The ring singularity, serious presentation



The ring singularity, not-so-serious-but-still-accurate presentation

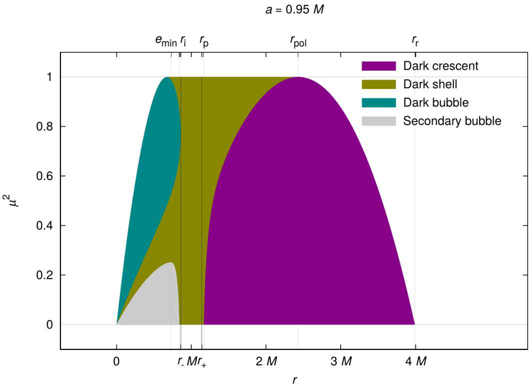
- **Warning !** Playing with the maximal analytic extension of the Kerr metric is **tricky** ! There are many issues regarding coordinate systems.
- A natural coordinate system is the Boyer-Lindquist one(t, r, θ, φ)
- It work outside the Kerr BH/wormhole, but NOT at horizon !
- When you cross the horizon, you need to switch to **ONE OF** the Kerr-Schild coordinate systems $(T, r, \theta, \tilde{\varphi})$

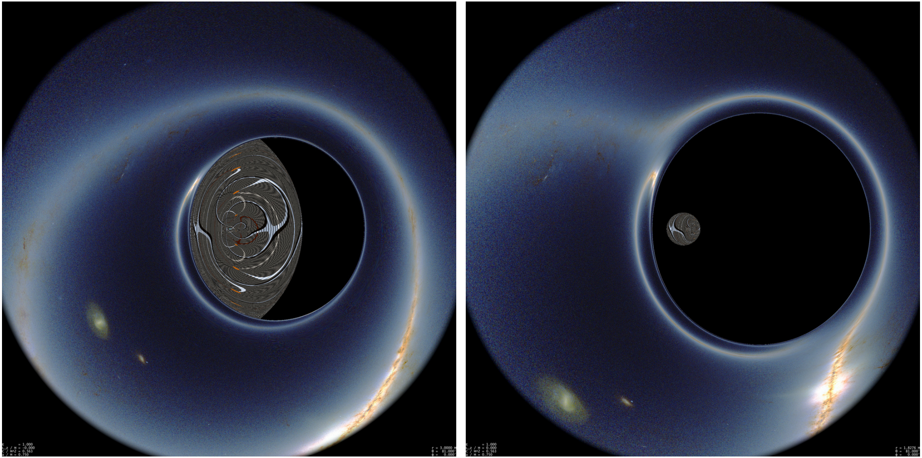
$$
\begin{aligned}
\dot{T} &\equiv \dot{t} + \epsilon \frac{2Mr}{\Delta} \dot{r}, \\
\dot{\tilde{\varphi}} &\equiv \dot{\varphi} + \epsilon \frac{a}{\Delta} \dot{r}, \\
\Delta &= r^2 - 2Mr + a^2, \\
\epsilon &= \pm 1.
\end{aligned}
$$



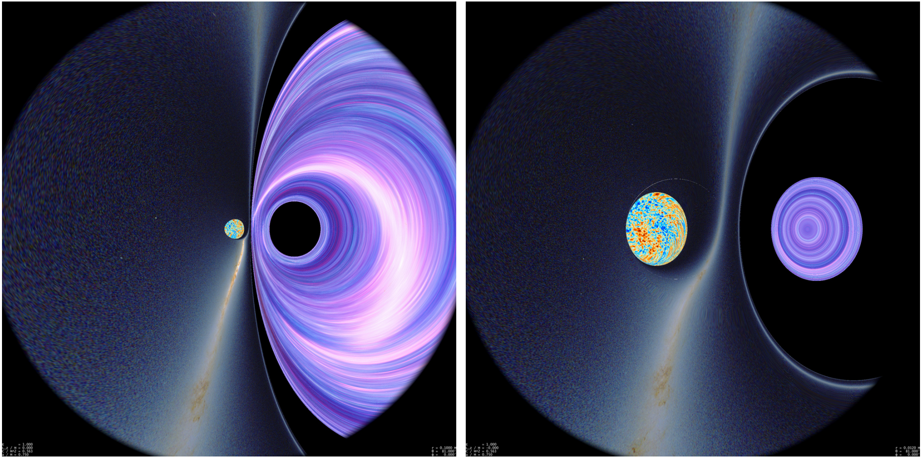
- The value of ϵ determines what is (incorrectly) called the "ingoing" or "outgoing" Kerr-Schild coordinates.
- In fact the choice of ϵ depends on whether the horizon you cross is parallel to the first or second bissector.
- Several types of null geodesics :
  - Flyby (either $r > r_{hor}$ or $r < 0$)
  - Transit ($r = +\infty$ to $r = -\infty$ or vice-versa)
  - Crossing (cross outer then inner horizon, bounces and crosses back the two horizons)
  - Adventuous (same as above but turning point at $r < 0$)
  - Bounded

Bounded null geodesics may intersect an observer's trajectory from $r = 4M$ down to $r = 0$. Its overvall shape may vary : crescent-shaped, ring-shaped or disc-shaped. Exact sequence depends on one's $r(\theta)$.

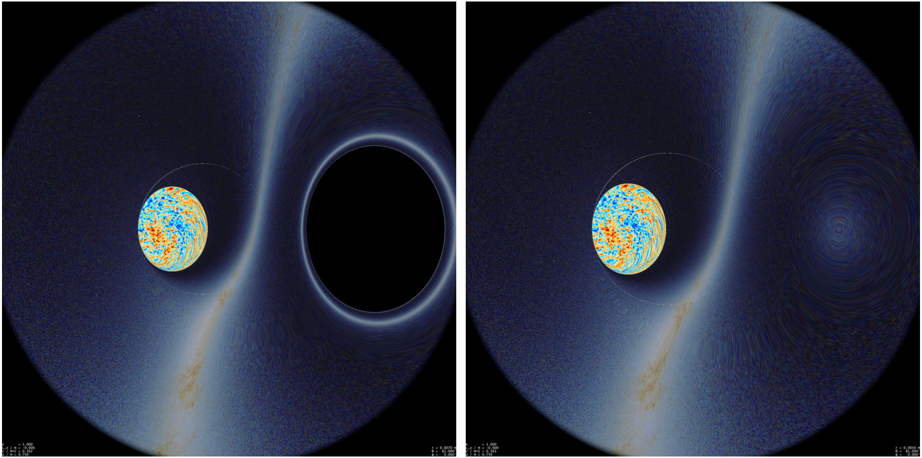




Bounded geodesics



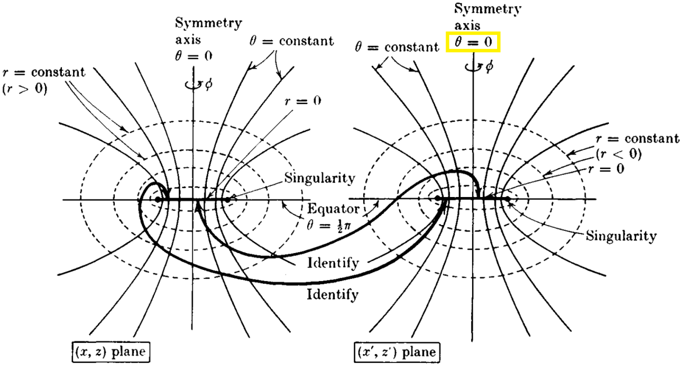
Bounded geodesics, continued



Bounded geodesics, continued

- Last but not least, we want (and we need) to cross the ring singularity.
- This is difficult because spheroidal Kerr-Schild coordinates are not regular at horizon crossing !
- We you cross the ring, you enter into a region where you flip the sign of r...
- But you also flip to orientation of the θ axis
- So θ is continuous at ring crossing, but $\dot{\theta}$ is not

This stuff is wrong ! ($\theta = 0$ should be either replaced by $\theta = \pi$ or put in the lower hemisphere)



So we need to switch to Cartesian Kerr-Schild coordinates (not fun)

$$
\begin{aligned}
x + iy &= (r + i\epsilon a) \sin\theta \;\exp(i\tilde{\varphi}), \\
z &= r \cos\theta
\end{aligned}
$$

r^2 is defined implicitely as

$$
r^4 - (x^2 + y^2 + z^2 - a^2) r^2 - a^2 z^2
$$

But this does not define r ! There are in fact **eight** coordinate systems here :

- $r > 0$
- $r < 0$
- You cross the ring "from above" starting from $r > 0$, i.e. $rz > 0$
- You cross the ring "from below" starting from $r > 0$, i.e. $rz < 0$
- You can choose ϵ as you like, this doubles the number

$$
\begin{aligned}
\ddot{x} + i\ddot{y} &= 4iMa\frac{r}{\Sigma^2}W\left[\dot{x} + i\dot{y} - \frac{x+iy}{r+i\epsilon a}\left\{\dot{r} - \epsilon a \sin^2\theta\dot{\tilde{\varphi}}\right.\right. \\
&\qquad \left.\left. + \left(\frac{4r^2}{\Sigma} - 3\right)\epsilon\frac{W^2-\kappa}{4W}\right\}\right] \\
&\quad -M(x+iy)\frac{r}{\Sigma^2}\left[\left(\frac{4r^2}{\Sigma} - 3\right)\kappa + \left(\frac{4r^2}{\Sigma} - 1\right)\frac{C - a^2W}{r(r+i\epsilon a)}\right], \\
\ddot{z} &= -Mz\frac{r}{\Sigma^2}\left[\left(\frac{4r^2}{\Sigma} - 3\right)\kappa + \left(\frac{4r^2}{\Sigma} - 1\right)\frac{C}{r^2}\right],
\end{aligned}
$$

with

$$
\begin{aligned}
\Sigma &= r^2 + a^2\cos^2\theta, \\
W &= \frac{C + \kappa r^2}{(r^2+a^2)E - aL_z - \epsilon\Sigma\dot{r}}.
\end{aligned}
$$

Ready ? Let us now go on the grand journey...