Colloquium to honor

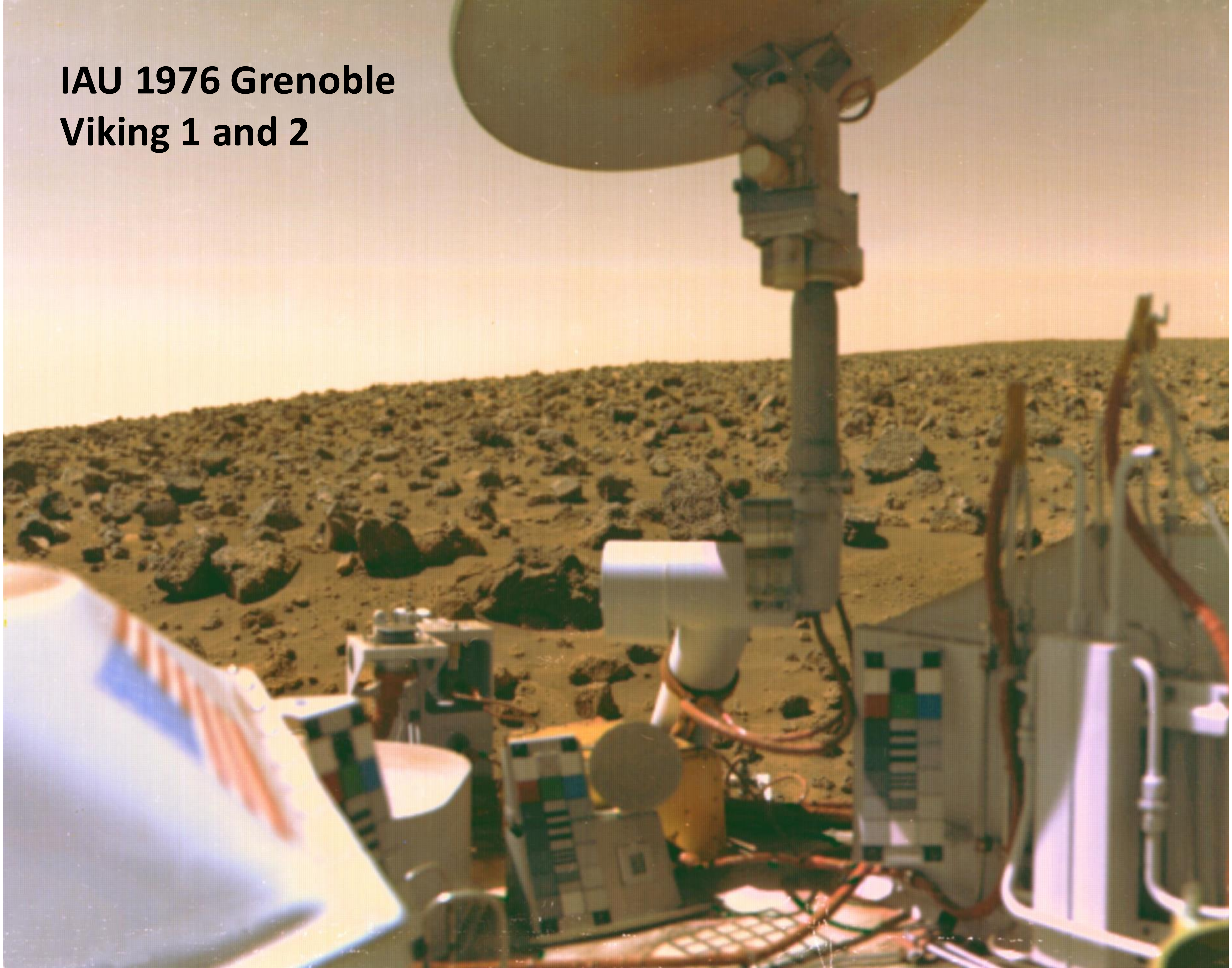
Prof. Sylvio Ferraz Mello

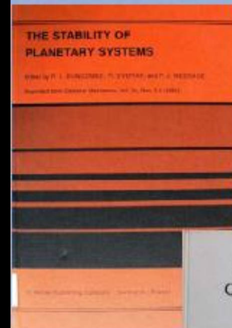
Paris observatory
8-10 march 2026

R.Dvorak, C. Lhotka and G. Pucacco

RESONANT CHAINS IN PLANETARY SYSTEMS

University of Vienna, University Roma Vergata


IAU 1976 Grenoble
Viking 1 and 2

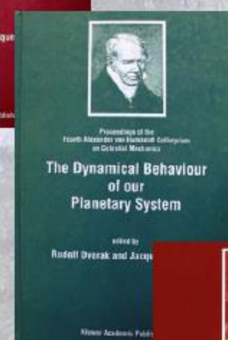


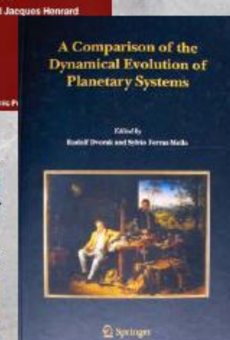
7. Alexander v. Humboldt Colloquium for Celestial Mechanics

universität wien

The chaotic dynamics of small bodies and planets

Bad Hofgastein, Salzburg, Austria
30.3.2008–5.4.2008
Hotel Winkler





SOC:

Chairman:
Sylvio Ferraz-Mello (BRAZIL)

Co-chairperson:
R. Dvorak and
E. Pilat-Lohinger (Austria)

T. Bountis (Greece)
A. Celletti (Italy)
P. Robutel (France)
D. Scheeres (USA)
K. Strassmeier (Germany)
Y.S. Sun (China)

LOC:

R. Dvorak
E. Pilat-Lohinger
B. Funk
C. Lhotka
R. Schwarz

TOPICS:

Stability of extrasolar planets
Chaotic motion of small bodies
Terrestrial planets in extrasolar systems
Satellite projects
Astrobiology

Invited Speakers:

G. Contopoulos (Greece), A. Celletti (Italy), A. Giorgilli (Italy), G. Gomez (Spain), D. Scheeres (USA), J. Laskar (France), F. Roig (Brazil), J.L. Zhou (China), C. Beauge (Argentina), K. Strassmeier (Germany)

FWF Der Wissenschaftsfonds

BMW_F Bundesministerium für Wissenschaft und Forschung

Chaos and secular variations of planar orbits in 2:1 resonance with Dione

S. Ferraz-Mello[1] and R. Dvorak[2]

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[2] Institut für Astronomie, Universität Wien, Türkenschanzstrasse 17, A-1180 Wien, Austria

Received August 11, accepted November 24, 1986

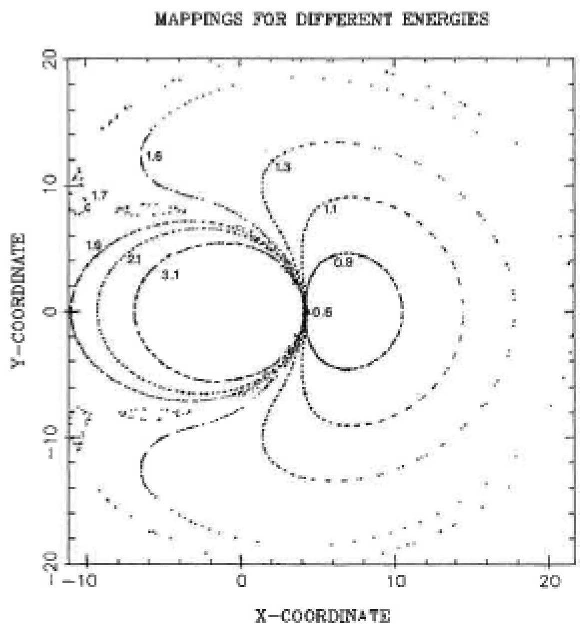


Fig. 4. Nine selected mappings starting from the same initial values of $x_0 = 4.155$, $\theta_0 = 0$ and $y_0 = 0$ for different energies from $H = 0.62311$ (actual value of Enceladus) to $H = 3.1$. Each energy level is represented by 200 intersections with the surface of section. Coordinates as in Fig. 1

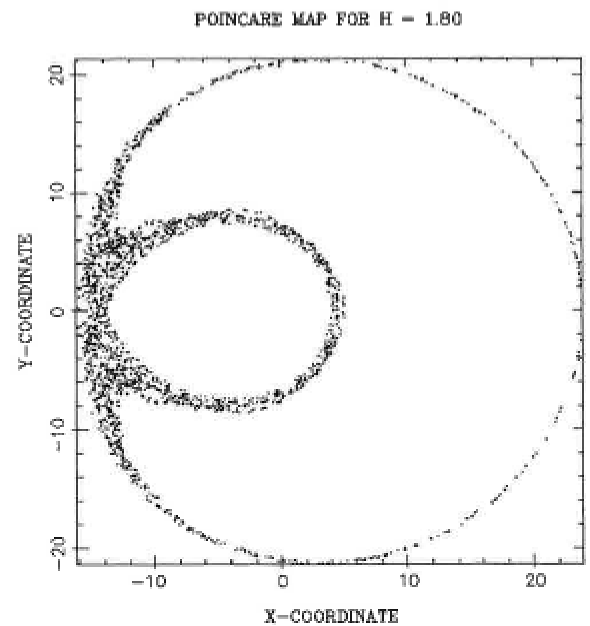
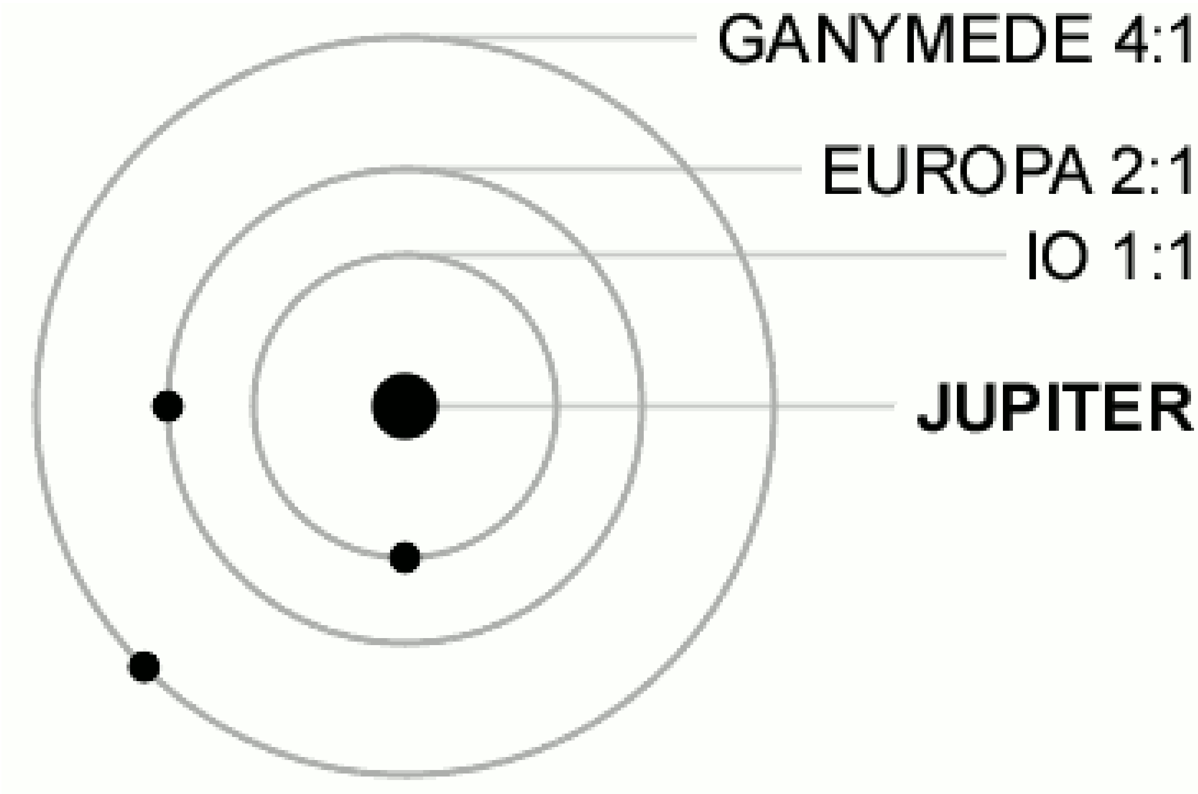


Fig. 6. Stochastic region at $H = 1.8$ and $x_0 = 4.155$ for 2000 intersections with the surface of section. Coordinates as in Fig. 1

OUTLINE

1. The Laplace resonance 4:2:1
2. The Jupiter Moons Io, Europa and Ganymede
3. Analytical description of the problem
4. Numerical exploration of the mass distribution
5. The results
6. Future prospects: resonances in extrasolar planetary systems


Kallisto
Ganymed
Io
Jupiter
Europa


GANYMEDE 4:1
EUROPA 2:1
IO 1:1
JUPITER


Ganymede: 7.2 days
Europa: 3.6 days
Io: 1.8 days
JUPITER

ANALYTIC PART: First order model

4-body problem: large central body + 3 relative small bodies in a plane

$a_j, e_j, \lambda_j, p_j, j = 1, 2, 3$

k_1=2 , k_2=2

2:1, 2:1 -> 4:2:1

LAPLACE RESONANCE

$$k_1 \overline{n}_2 = (k_1 - 1)\overline{n}_1$$
$$k_2 \overline{n}_3 = (k_2 - 1)\overline{n}_2.$$

$$j_1 \dot{\lambda}_1 + j_2 \dot{\lambda}_2 + j_3 \dot{\lambda}_3 \simeq 0, \quad j_1, j_2, j_3 \in \mathbb{Z},$$

$$H(L_j, P_j, \lambda_j, p_j) = H_{kep} + H_{res}$$

Delaunay variables

$$H_{kep} = \sum_{i=1}^{3} \left(\overline{n}_i (L_i - \overline{L}_i) - \frac{3}{2} \eta_i (L_i - \overline{L}_i)^2 \right)$$

$$M_1 = m_0 + m_1 \qquad \mu_1 = \frac{m_0 m_1}{M_1}$$

$$\overline{n}_i \doteq \sqrt{\frac{M_i}{\overline{a}_i{}^3}} = \frac{M_i{}^2 \mu_i{}^3}{\overline{L}_i{}^3}$$

$$H_{kep} = -\frac{1}{2} \sum_{k=1}^{3} \frac{M_k{}^2 \mu_k{}^3}{L_k{}^2}$$

$$\eta_i \doteq \frac{\overline{n}_i}{\overline{L}_i}.$$

mean motions

- Celletti, Paita, Pucacco (2019),
- Pucacco, CMDA 133 (2021),
- Celletti et al., A&A 655 (2021), R&CD 27 (2022),
- Pucacco, CMDA 136 (2024).

$$
\begin{aligned}
H_{res} = & -\alpha\sqrt{2P_1}\cos(k_1\lambda_2 - (k_1-1)\lambda_1 + p_1) \\
& -\beta_1\sqrt{2P_2}\cos(k_1\lambda_2 - (k_1-1)\lambda_1 + p_2)) \\
& -\beta_2\sqrt{2P_2}\cos(k_2\lambda_3 - (k_2-1)\lambda_2 + p_2) \\
& -\gamma\sqrt{2P_3}\cos(k_2\lambda_3 - (k_2-1)\lambda_2 + p_3)),
\end{aligned}
$$

$\alpha, \beta_1, \beta_2, \gamma$

Functions of the masses and the Laplace coefficients

For the location of the equilibria and their stability a new set of canonical variables adapted to the 1st order resonance is introduced

$$(L_j, P_j, \lambda_j, p_j),\ j = 1, 2, 3 \longrightarrow (Q_\alpha, q_\alpha),\ \alpha = 1, \ldots, 6$$

$$(L_j, P_j, \lambda_j, p_j),\ \ j = 1, 2, 3 \longrightarrow (Q_\alpha, q_\alpha),\ \ \alpha = 1, \ldots, 6$$

$$\begin{aligned}
Q_j &= P_j \ \ j = 1, 2, 3 \\
Q_4 &= \frac{k_1}{3(k_1 - 1)} L_1 + \frac{L_2}{3} + \frac{L_3}{3}\left(1 - \frac{4}{k_2}\right) \\
Q_5 &= \frac{k_1 + k_2 - 1}{3(k_1 - 1)} L_1 + \frac{L_2}{3} + \frac{1}{3}(k_2 - 1)(P_1 + P_2 + P_3) \\
Q_6 &= L_1 + L_2 + L_3 - P_1 - P_2 - P_3
\end{aligned}$$

and

$$\begin{aligned}
q_1 &= k_1 \lambda_2 - (k_1 - 1)\lambda_1 + p_1 \\
q_2 &= k_1 \lambda_2 - (k_1 - 1)\lambda_1 + p_2 \\
q_3 &= k_1 \lambda_2 - (k_1 - 1)\lambda_1 + p_3 \\
q_4 &= (k_1 \lambda_2 - (k_1 - 1)\lambda_1) - (k_2 \lambda_3 - (k_2 - 1)\lambda_2) \\
&= (1 - k_1)\lambda_1 + (k_1 + k_2 - 1)\lambda_2 - k_2 \lambda_3 \\
q_5 &= \frac{4(k_1 - 1)}{k_2} \lambda_1 - \frac{4(k_1 - 1) + k_2}{k_2} \lambda_2 + \lambda_3 \\
q_6 &= \frac{(k_1 - 1)(k_2 - 4)}{3k_2} \lambda_1 \\
&\quad - \frac{(k_2 - 4)(k_1 + k_2 - 1)}{3k_2} \lambda_2 + \frac{1}{3}(k_2 - 1)\lambda_3.
\end{aligned}$$

$$H(Q_a, q_a; Q_5, Q_6) = \sum_{n=0}^{2} H_n(Q_a, q_a), \quad a = 1, ..., 4.$$

angular momentum deficit $\Gamma = Q_1 + Q_2 + Q_3$

$$H=\omega\Gamma-3/2A\Gamma^2-3B\Lambda\Gamma-3/2C\Lambda^2+H_{res}(Q_j,q_j,\lambda)$$

spacing parameter $Q_L = 3Q_5 + (k_2 - 1)Q_6$

total angular momentum $Q_6 = \sum_j (L_j - Q_j)$

$$H = \omega\Gamma - \frac{3}{2}A\Gamma^2 - 3B\Lambda\Gamma - \frac{3}{2}C\Lambda^2 + H_{res}(Q_j, q_j, \lambda).$$

$\Gamma = Q_1 + Q_2 + Q_3$ *angular momentum deficit*

A,B and C are functions of mean motion and masses

Resonance Proximity Parameter

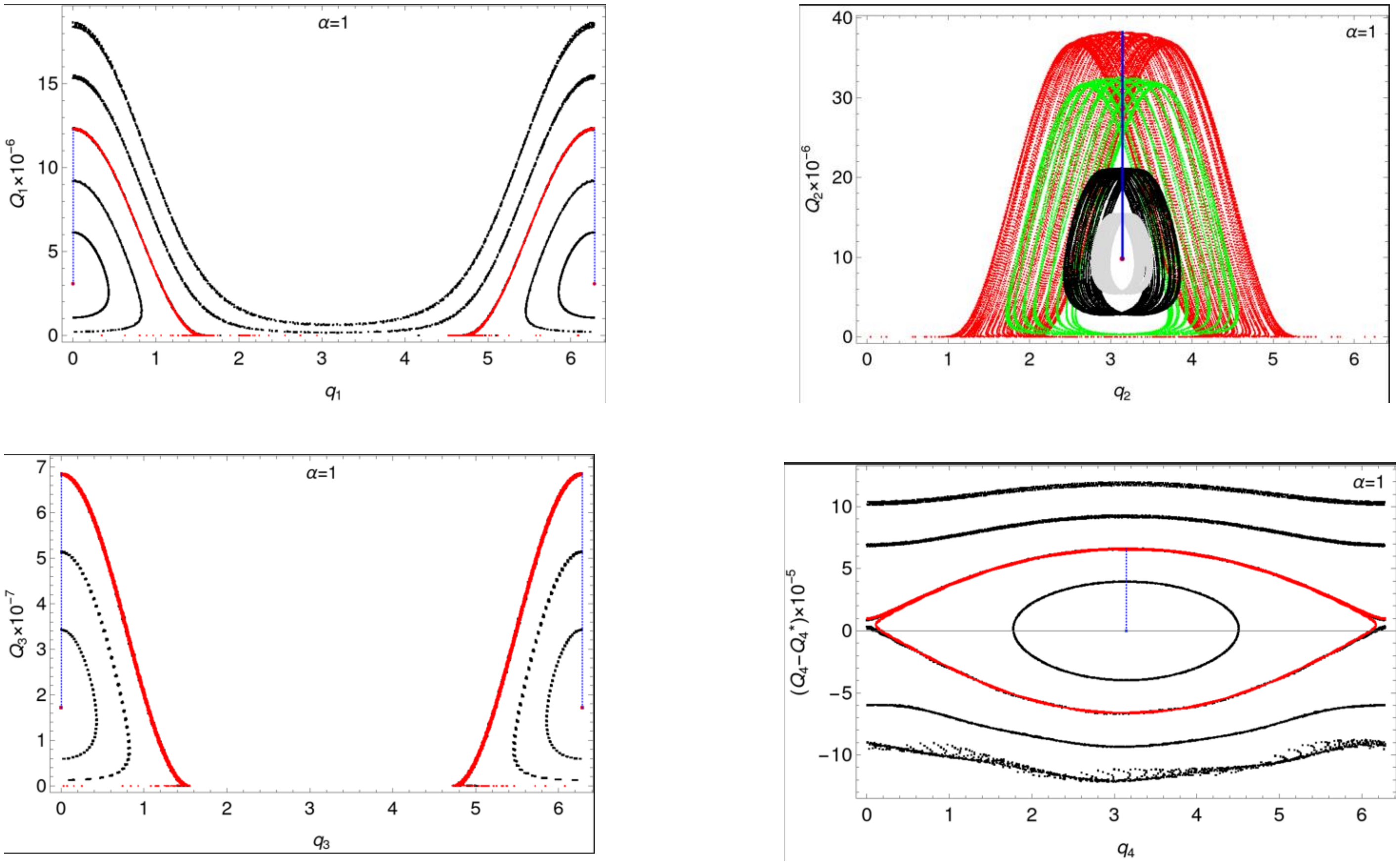
$$\omega = \frac{1}{C}\left[3(AC - B^2)\Gamma - (C - B)\Delta_{12} - B\Delta_{23}\right],$$

where

$$\Delta_{12} = (k_1 - 1)n_1 - k_1 n_2, \quad \Delta_{23} = (k_2 - 1)n_2 - k_2 n_3.$$

Phase space topology

still 8 dimensional



The dynamical model is the complete plane 4body problem of the Sun + 3 massive planets

Find the ‘correct’ initial conditions with the results of the 1st order theory for given masses m1, m2 and m3 for a1…a3 and e1..e3 which lead to a Laplace resonance 4:2:1

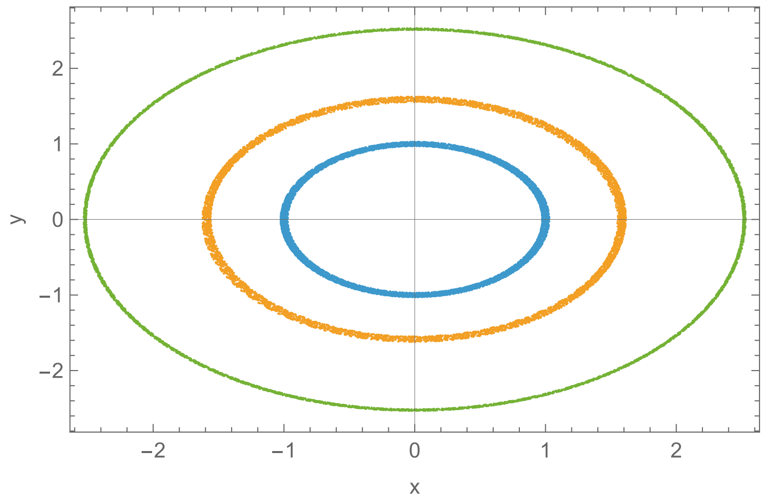
Integrate with these values the original equations of motion

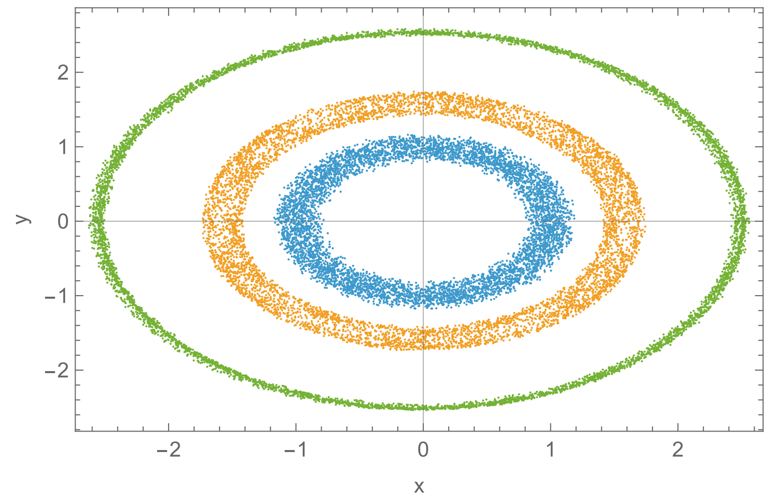
Use a numerical integrator with a high precision for up to 10 kyrs (LIE-integrator)

Do this with mass ratios m_i/M (i=1,2,3) = 0.001, 0.0001 and 0.00001
For m_i (i=1,9) for all possibilities of m1:m2:m3
e.g. for m/M = 0.0001
m1-m2-m3 = 1:1:1 m1=0.0001,m2=0.0001 ,m3=0.0001
m1-m2-m3 = 3:1:8 m1=0.0003, m2=0.0001 , m3=0.0008

check the Laplace stability over this time interval

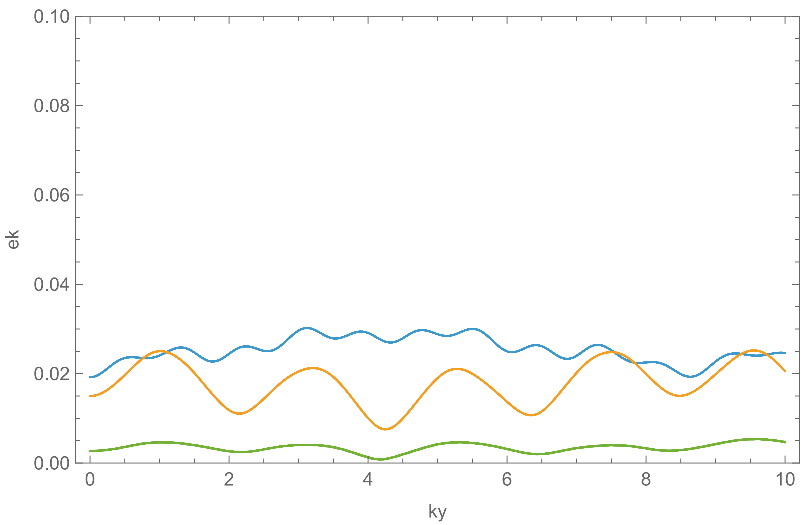
→ These integrations provide the necessary ratio m1-m2-m3 which lead to
→ stable configurations in the Laplace sense
→ so that the ratios in mean motion resonance are close to 4:2:1

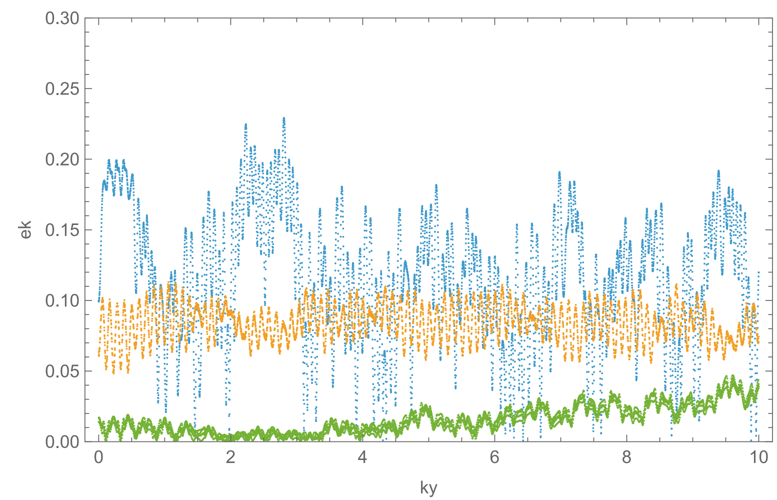




resonant stable

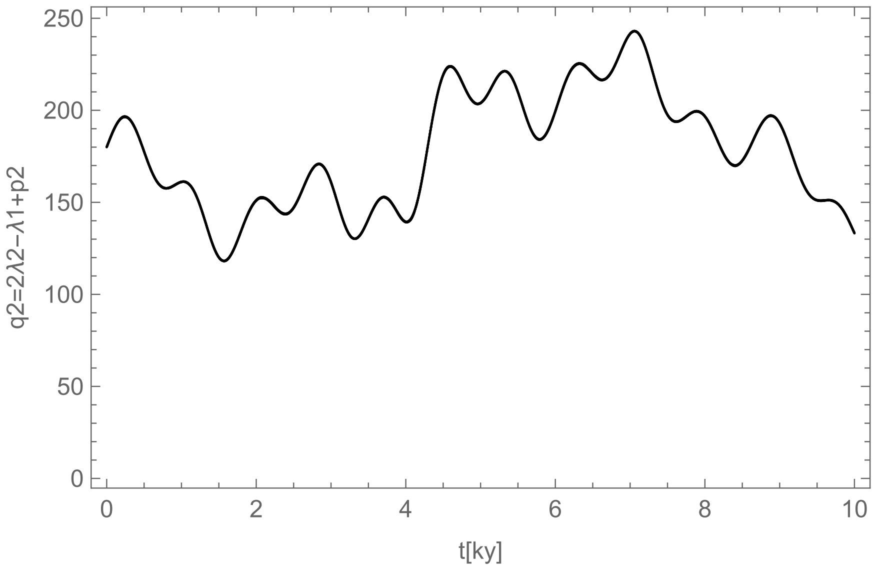
unstable



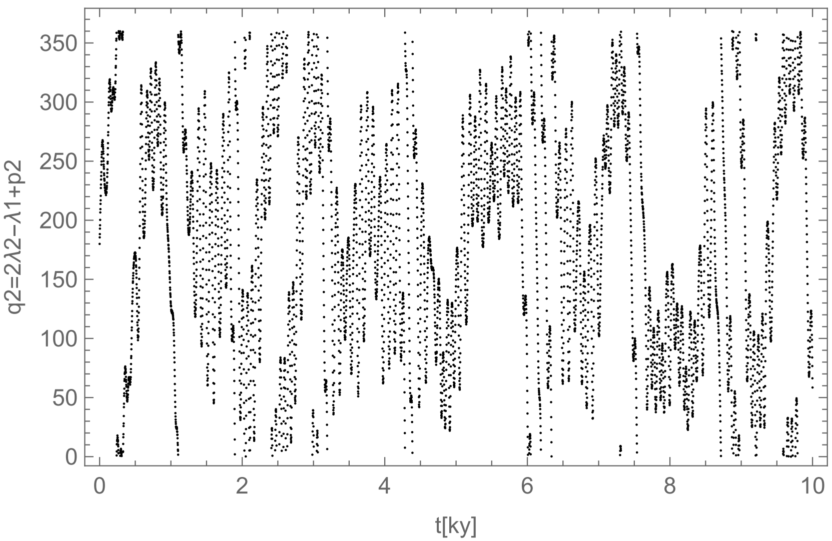


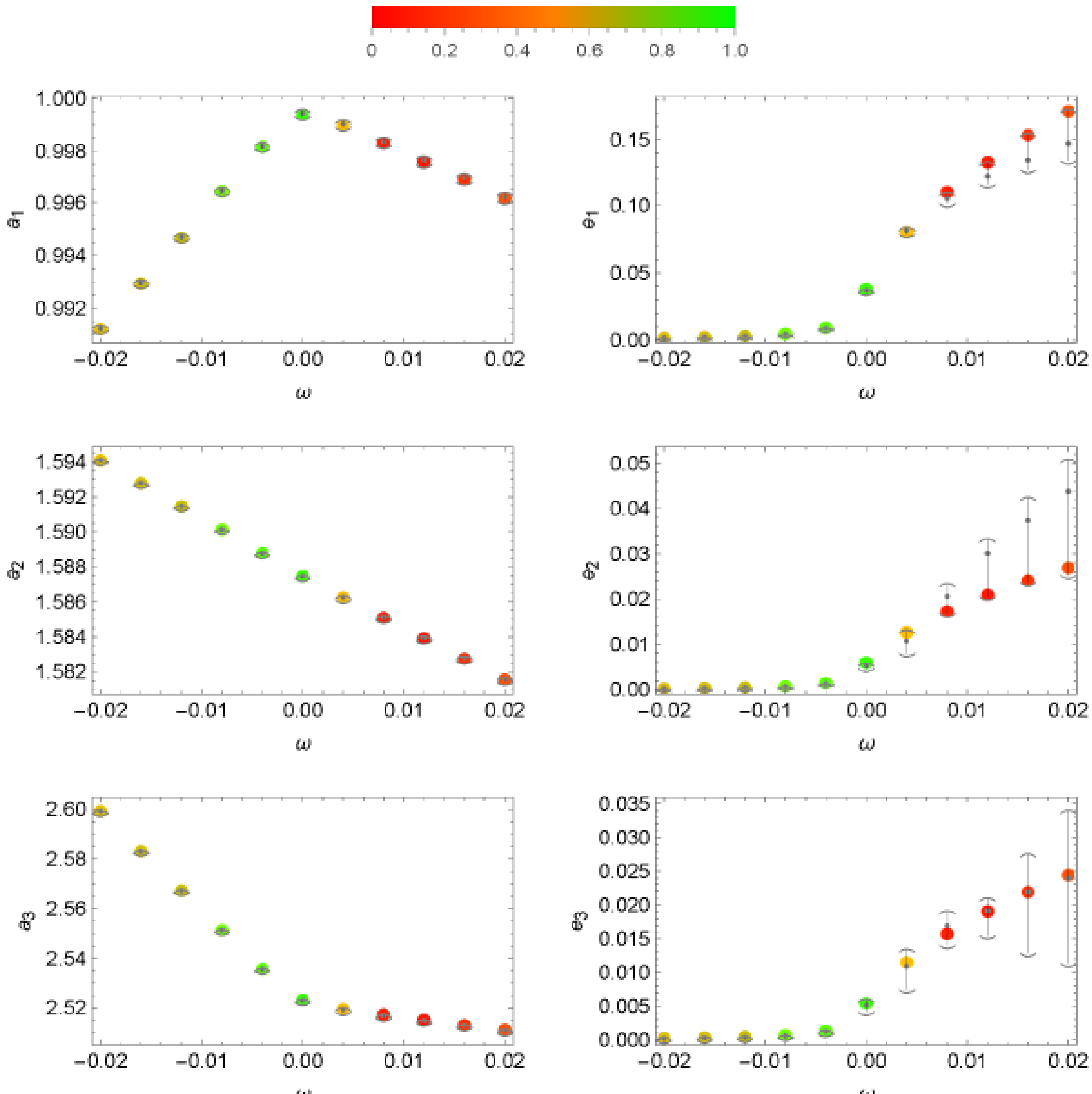
LIBRATION INDEX

resonant stable



unstable





m1=10^-5

m2=5.10^-5

m3=10^-5

m1=m2=m3

10^-5
<m_k
<10^-3

k=1,2,3

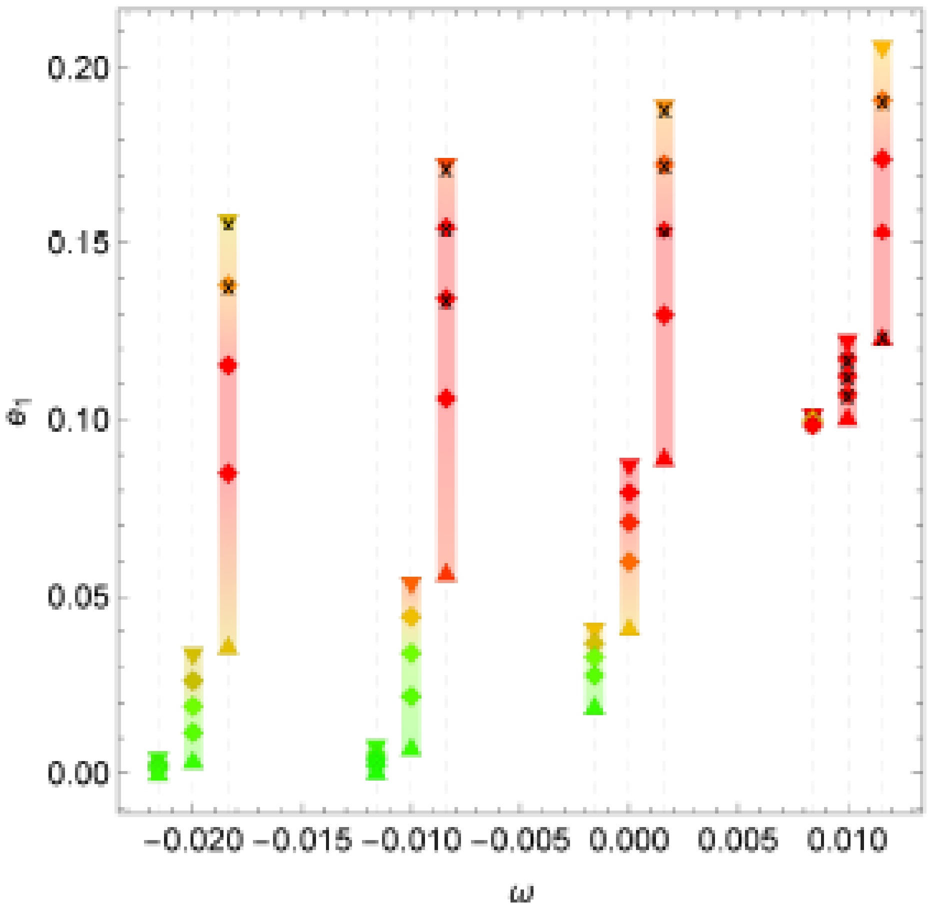


Fig. 3. Influence of variations in magnitude of mass on the libration index i_4 (standard color code) and resonant values of e_1 for different resonance proximity parameter ω.

10^-5 left

10^-4 middle

10^-3 right

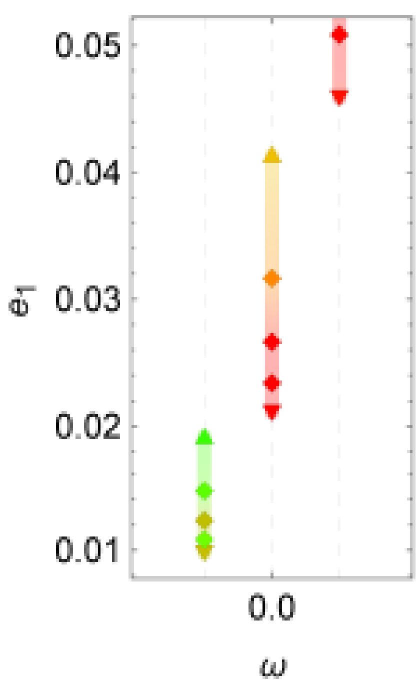
m_k=1,3,5,7,9

Symbols:

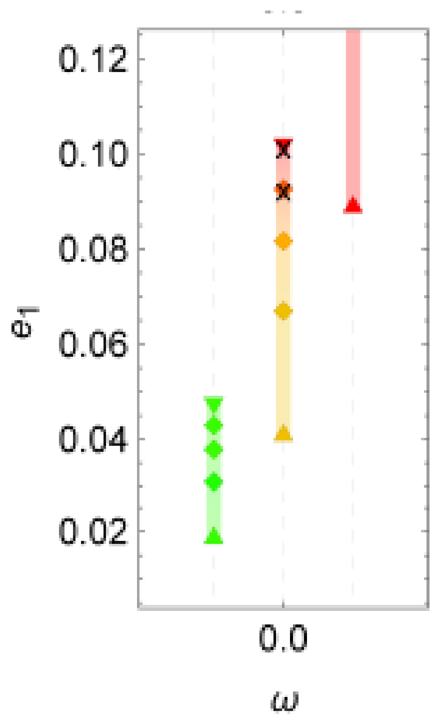
upside triangle
downside triangle
diamonds (3,5,7)

Influence of variations in magnitude of the j-th planet
the resonant value of the eccentricity e_k
For the k-th planet (k-th row)

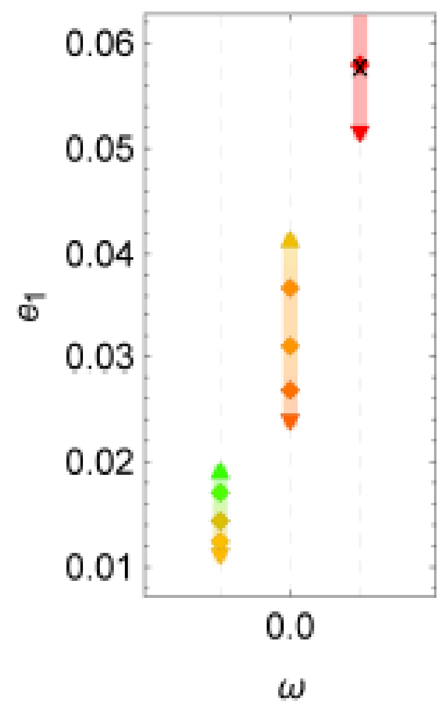
m_1



m_2

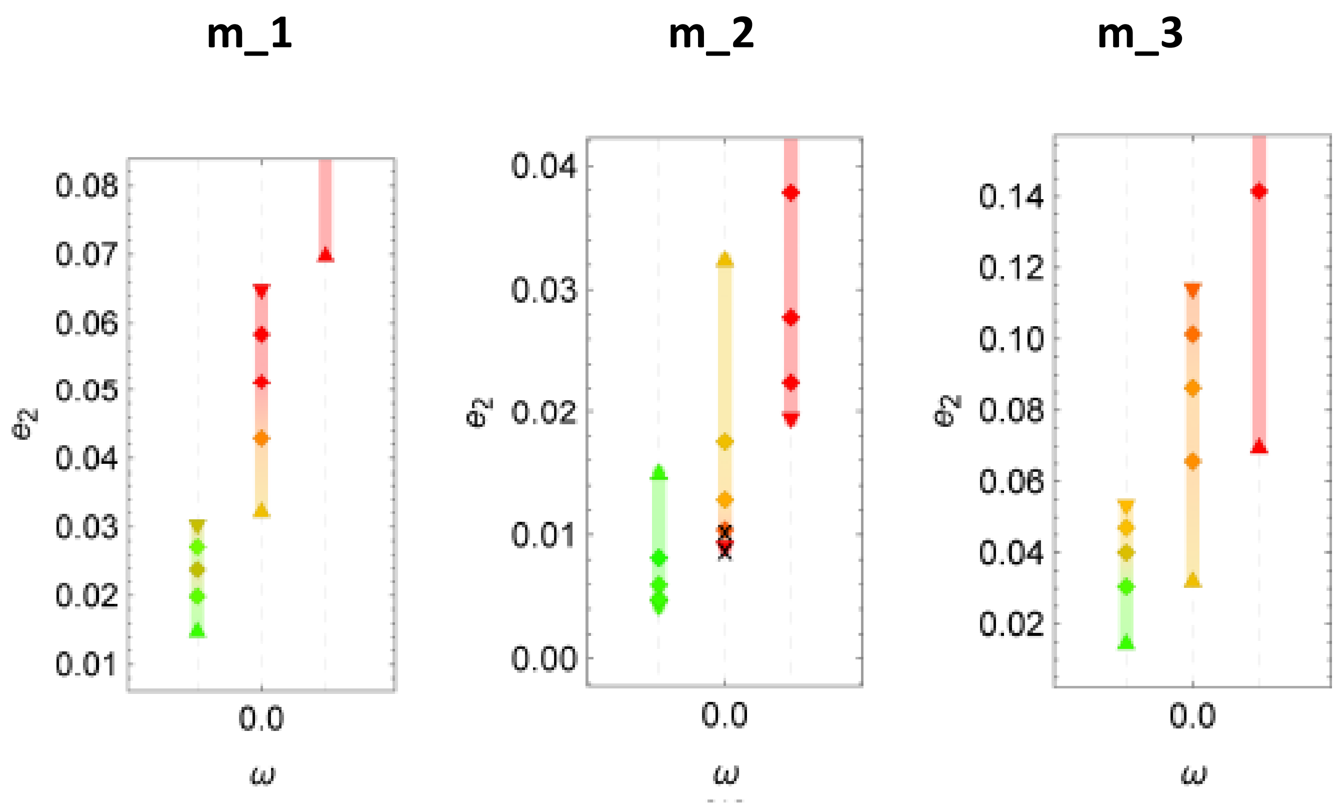


m_3



increasing m_1 -> more circular, m_2 -> more elliptic, m_3 ->more circular

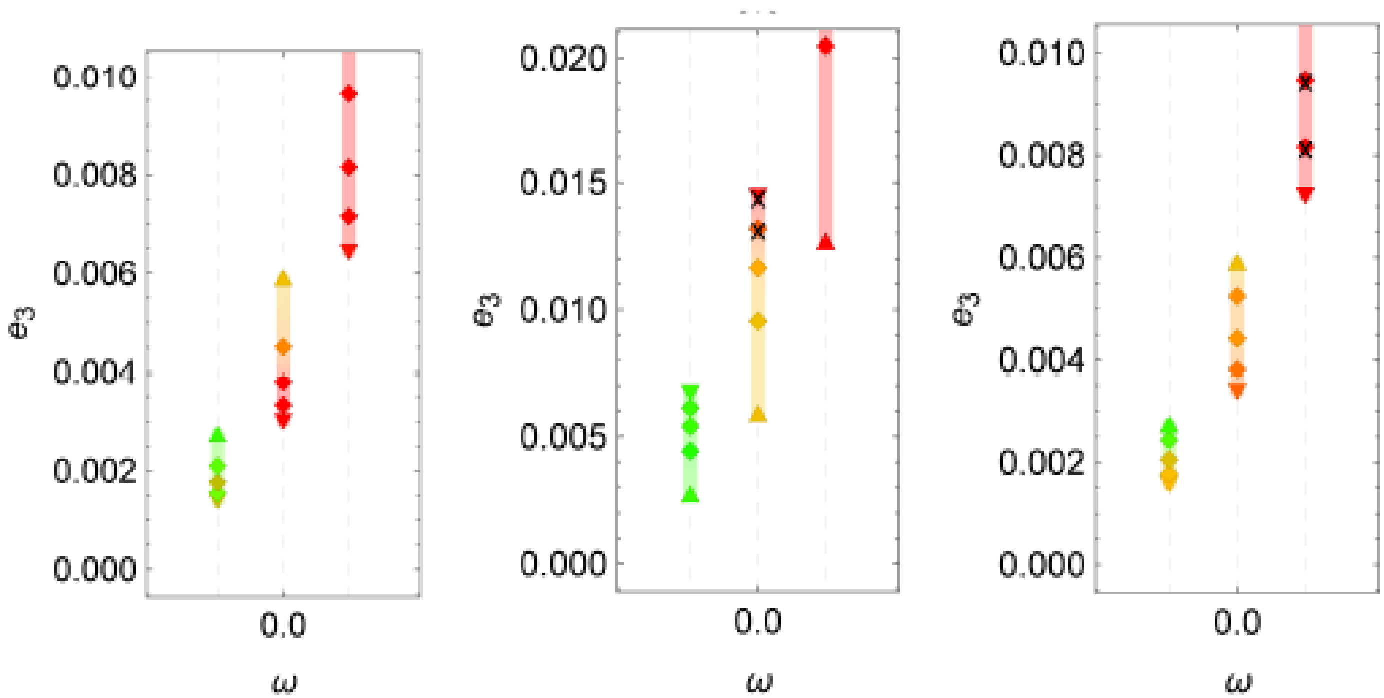
Influence of variations in magnitude of the j-th planet
the resonant value of the eccentricity e_k
For the k-th planet (k-th row) **k=2**



increasing m_1 -> more elliptic, m_2 -> more circular, m_3 ->more elliptic

Influence of variations in magnitude of the j-th planet
the resonant value of the eccentricity e_k
For the k-th planet (k-th row) **k=3**

m_1 **m_2** **m_3**

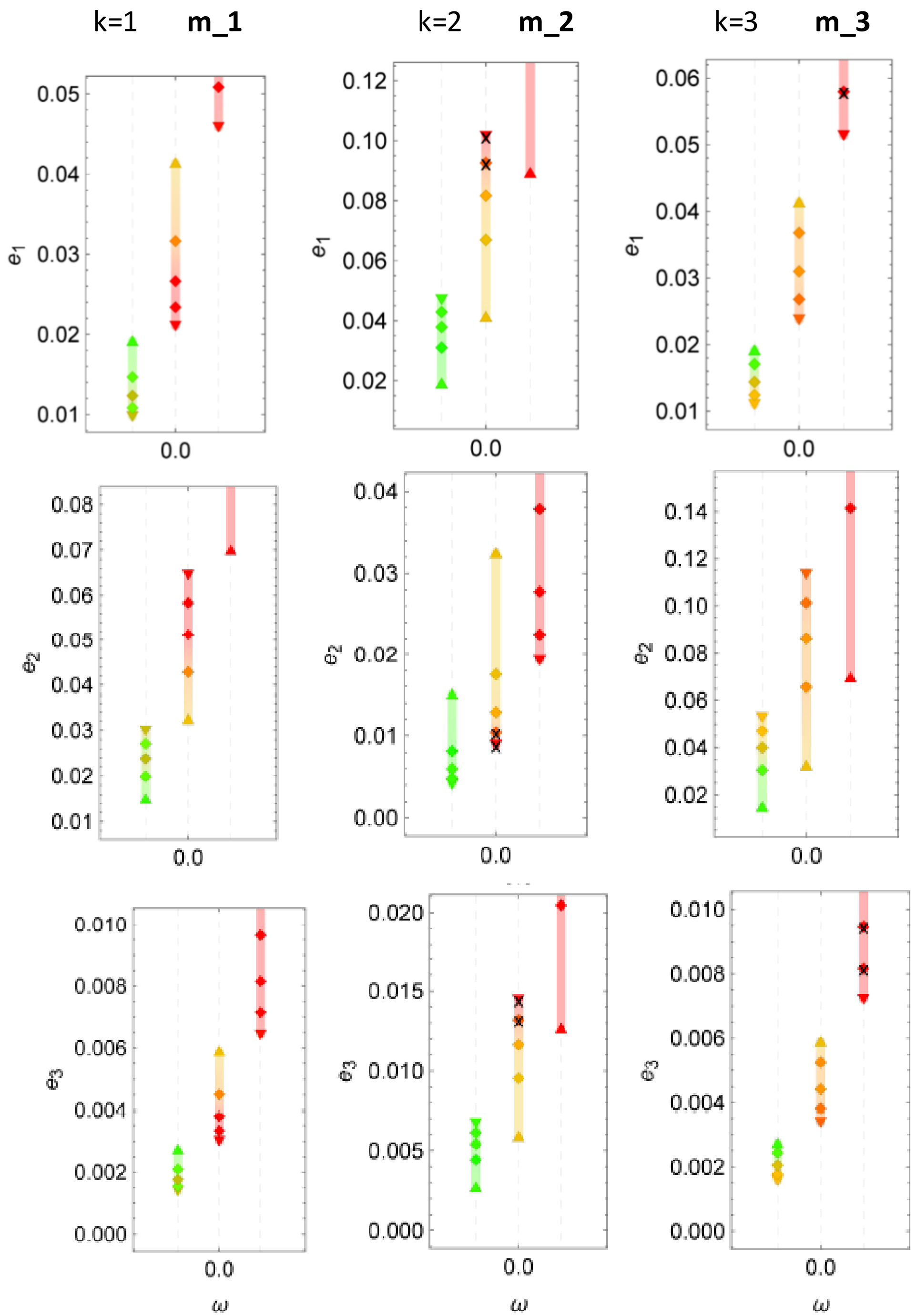


increasing m_1 -> more circular, m_2 -> more elliptc, m_3 ->more circular

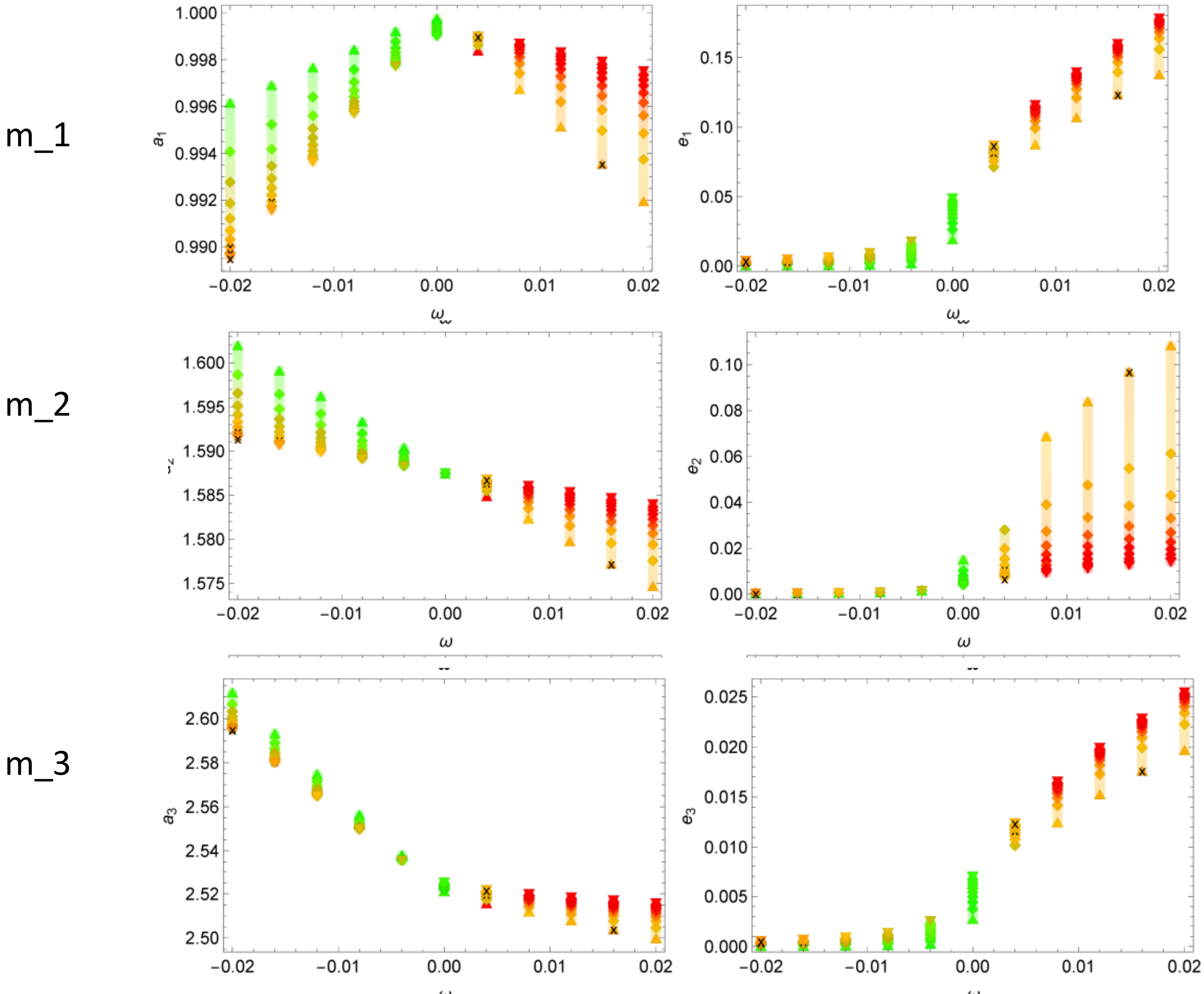
Influence of a different mass of the planet on the orbital parameters e_k on the innermost, the middle one and the outermost. Note that only for the smallest masses 10^-5 (left row) all the integrated orbits lead to stable Laplace resonances (green)

10^-5 left,10^-4 middle 10^-3 right column for m_k=1,3,5,7,9

Symbols:
upside triangle
downside triangle
diamonds (3,5,7)



Influence of variations in m_2 on a_k left diagram, e_k right diagram for m_2/10^-5 for a bigger mass[1,9], m_1=m_2=10^--5, omega [-0.02,0.02]



The Goal of this research was to find out the necessary parameters to ensure the stability of the Laplace configuration 2:1-2:1

1.Developing a linear theory using a first order averaged Hamiltonian model depending on the ‘resonance proximity parameter omega
2.Using this theory it was possible tocheck the linear stability and to compute initial conditions valid for the full problem 2:1-2:1
3.Integrating thousands of possible solutions, depending primarily on the three masses of the involved planets, up to million years
4. Analyzing the big amount of data with respect to the semi-major axes and eccentricities and their Laplace stability depending on omega and how they depend on the three masses involved via appropriate figures
5. Determine the Libration behaviour of the main resonant argument which turned out to be stable in the resonance chain 2:1-2:1 up to ratios of planetary masses 10^-4 in the real case while linear stability is guaranteed to the threshold 10^-3
6. The dominanting mass for instability is the middle mass

7 Outlook: Developing the theory for other resonant chains and checking the data of observations of exoplanetary system concerning long term stability .

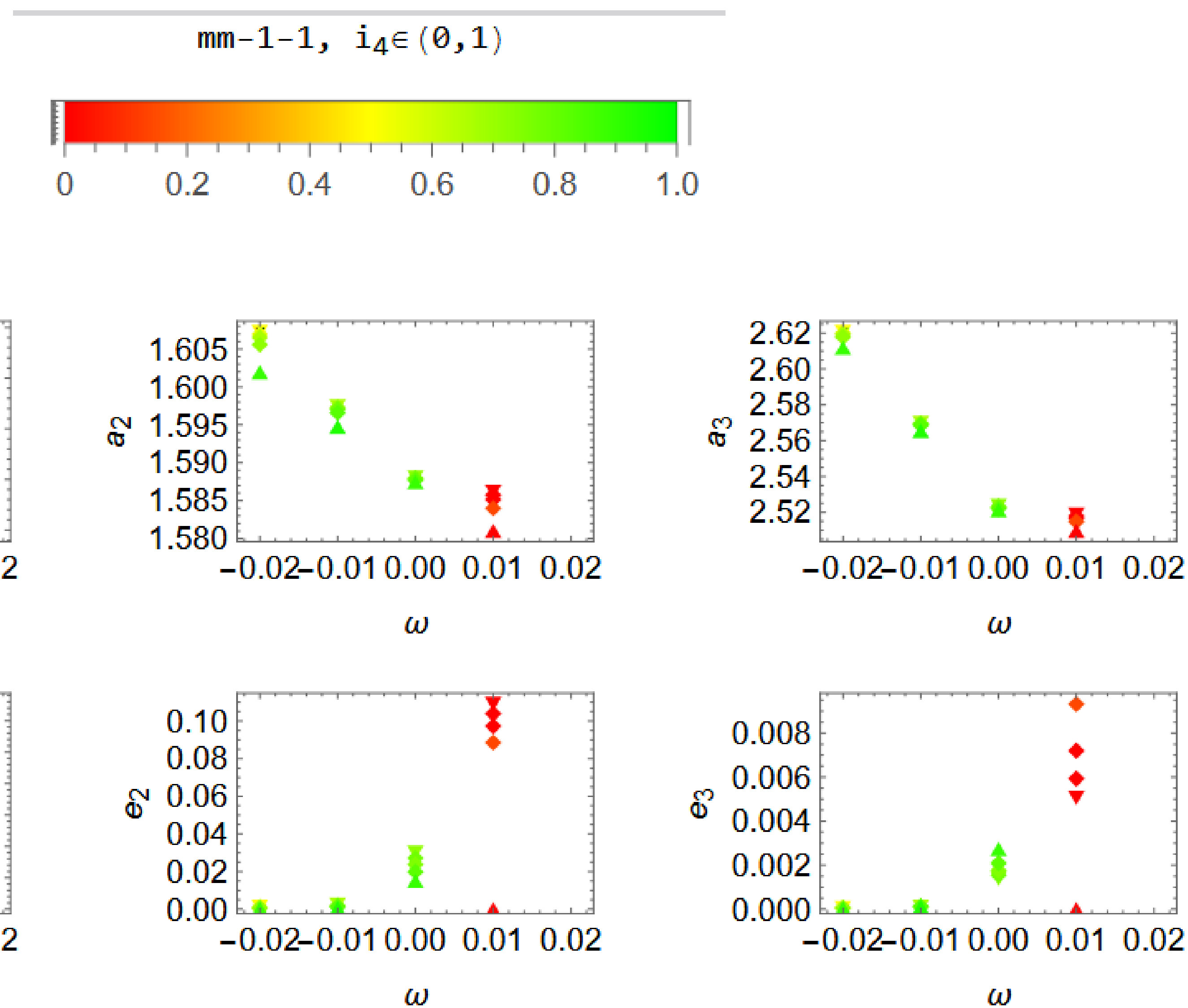
System	Bodies	Resonance	m	ω
Galilean system	Io-Europa-Ganymede	(2:1 – 2:1)	1	
GJ-876	planets c-b-e	(2:1 – 2:1)	1	
HR 8799	planets d-c-b	(2:1 – 2:1)	1	
Kepler 31	planets b-c-d	(2:1 – 2:1)	1	
GJ-2046 (HD40307)	planets b-c-d	(2:1 – 2:1)	1	
HIP 41378	planets b-c-g	(2:1 – 2:1)	1	
Kepler 114		(3:2 – 3:2)	2	
Kepler 305		(3:2 – 2:1)	1	
Kepler 326		(2:1 – 3:2)	2	
YZ Cet		(3:2 – 3:2)	2	

To appear soon in Astronomy & Astrophysics

On the stability of first-order mean motion resonant chains in planetary systems

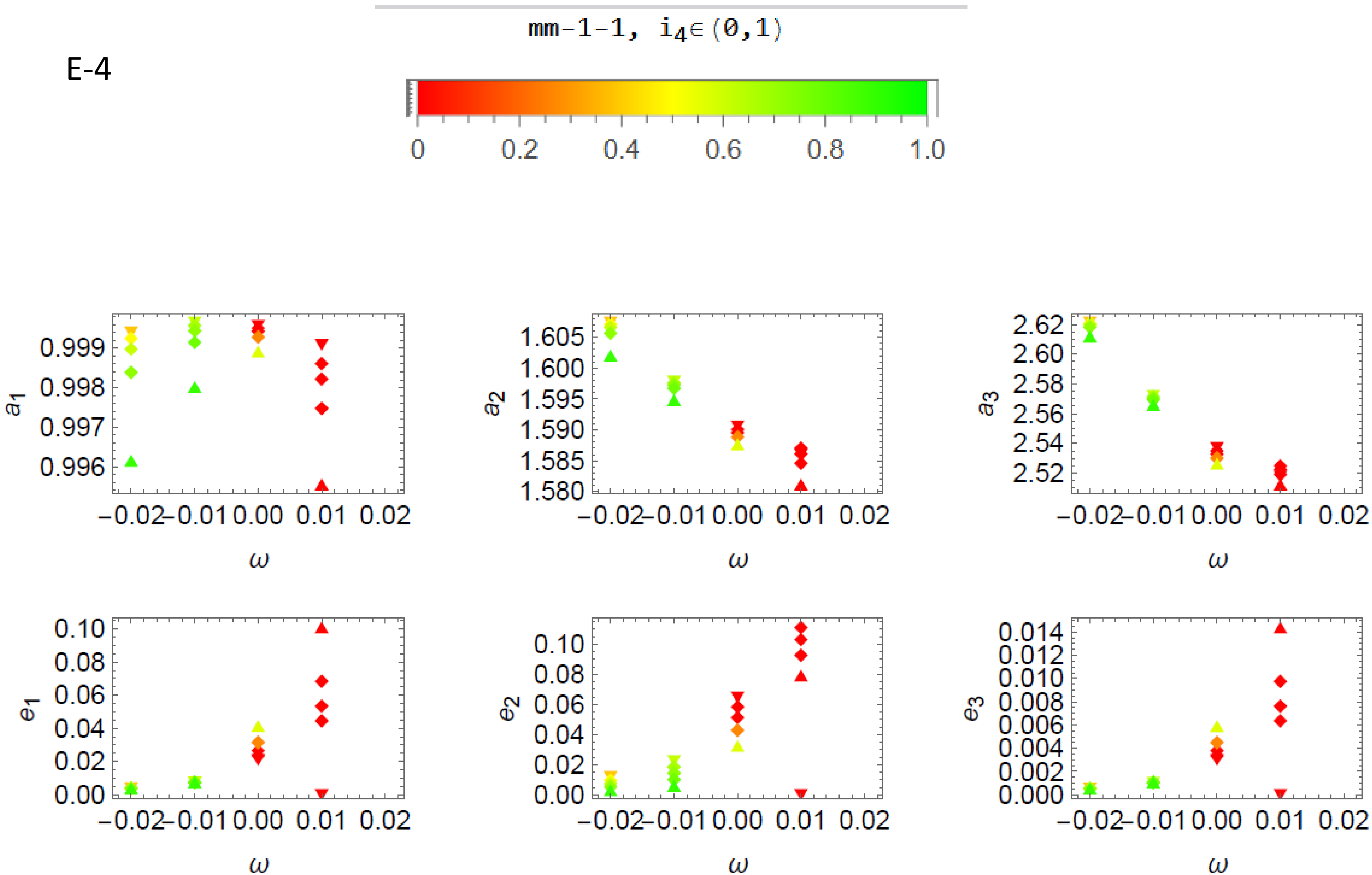
Christoph Lhotka, Giuseppe Pucacco, Rudolf Dvorak

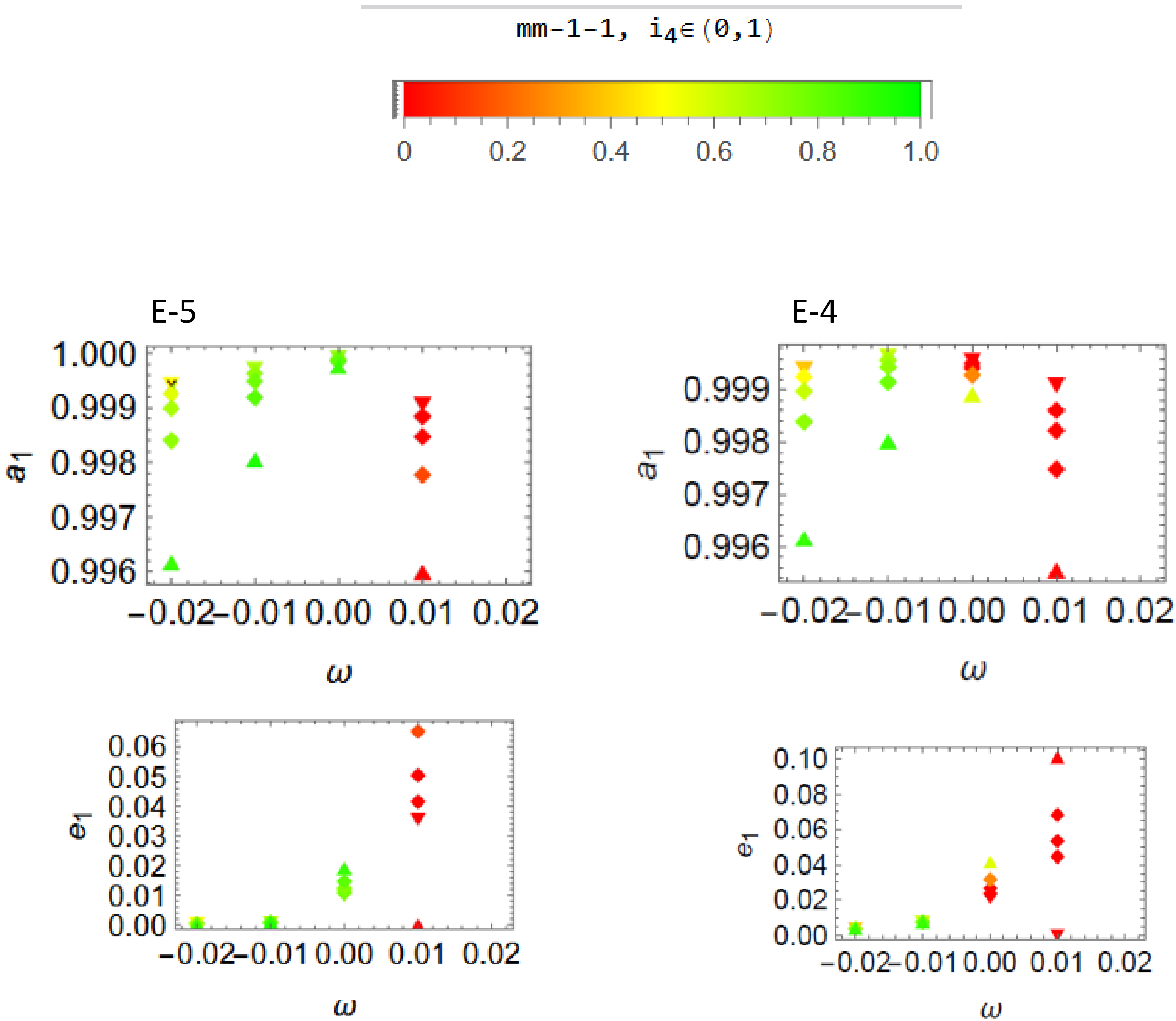
E-5



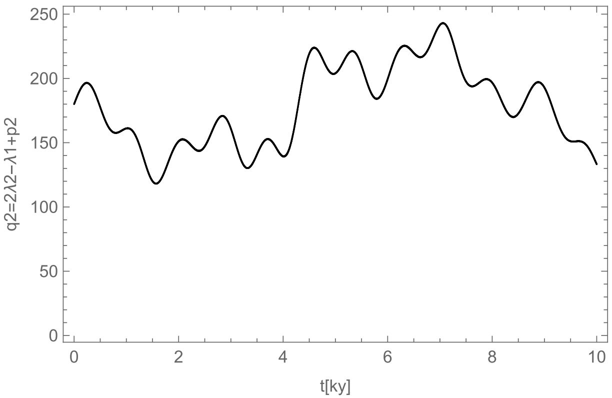
mm-1-1, $i_4 \in (0,1)$
0
0.2
0.4
0.6
0.8
1.0
a_1
1.000
0.999
0.998
0.997
0.996
a_2
1.605
1.600
1.595
1.590
1.585
1.580
a_3
2.62
2.60
2.58
2.56
2.54
2.52
e_1
0.06
0.05
0.04
0.03
0.02
0.01
0.00
e_2
0.10
0.08
0.06
0.04
0.02
0.00
e_3
0.008
0.006
0.004
0.002
0.000
−0.02
−0.01
0.00
0.01
0.02
ω

E-4

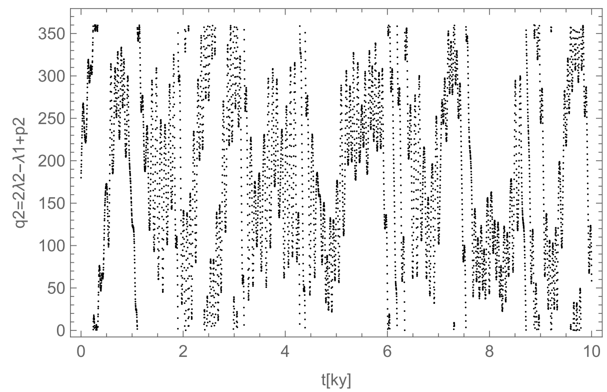

mm-1-1, $i_4 \in (0,1)$
0
0.2
0.4
0.6
0.8
1.0
a_1
0.999
0.998
0.997
0.996
a_2
1.605
1.600
1.595
1.590
1.585
1.580
a_3
2.62
2.60
2.58
2.56
2.54
2.52
e_1
0.10
0.08
0.06
0.04
0.02
0.00
e_2
0.10
0.08
0.06
0.04
0.02
0.00
e_3
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
−0.02 −0.01 0.00 0.01 0.02
ω



mm-1-1, $i_4 \in (0,1)$
0
0.2
0.4
0.6
0.8
1.0
E-5
E-4
a_1
e_1
ω
1.000
0.999
0.998
0.997
0.996
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0.10
0.08
−0.02
−0.01
0.00
0.01
0.02

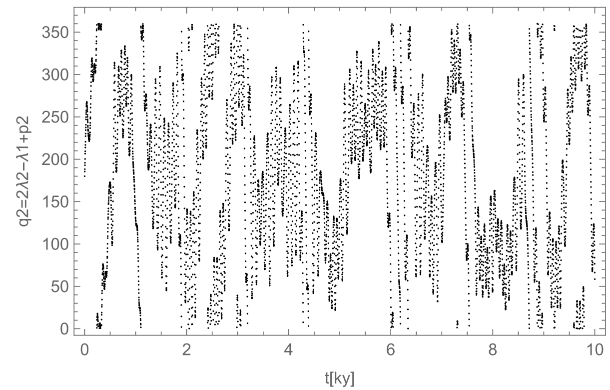


resonant stable



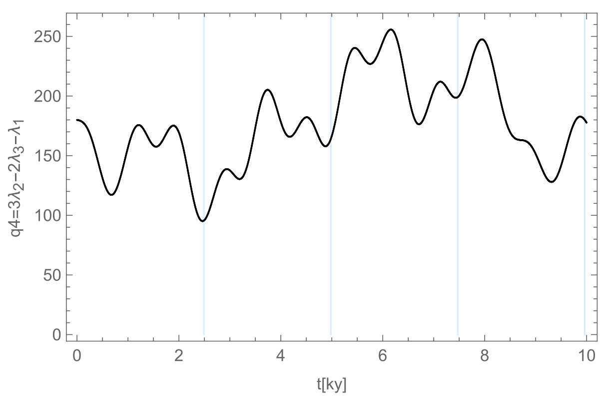
unstable

Long term stability (2/2)



unstable

resonant stable



Some references:

- Dvorak, Hanslmeier, Lichtenegger BAS 15 (1983),
- Hanslmeier, Dvorak, A&A 132 (1984),
- Eggl, Dvorak LNP 790 (2010),
- Dvorak, Cuntz, AN 345 (2024).

$$a_j, e_j, \lambda_j, p_j, j = 1, 2, 3$$

$$k_1 \overline{n}_2 = (k_1 - 1)\overline{n}_1, \quad k_2 \overline{n}_3 = (k_2 - 1)\overline{n}_2,$$

$$H(L_j, P_j, \lambda_j, p_j) = H_{kep} + H_{res}$$

$$\begin{aligned} H_{res} = &- \alpha \sqrt{2P_1} \cos(k_1 \lambda_2 - (k_1 - 1)\lambda_1 + p_1) \\ &- \beta_1 \sqrt{2P_2} \cos(k_1 \lambda_2 - (k_1 - 1)\lambda_1 + p_2)) \\ &- \beta_2 \sqrt{2P_2} \cos(k_2 \lambda_3 - (k_2 - 1)\lambda_2 + p_2) \\ &- \gamma \sqrt{2P_3} \cos(k_2 \lambda_3 - (k_2 - 1)\lambda_2 + p_3)), \end{aligned}$$

Figure 2: show 11 different initial conditions leading to an equilibrium point in the Laplace sense computed with the aid of the linear theory for 100 kyr for fixed masses $m_1=m_3=10^{-5}$ and $m_2= 5.10^{-5}$ in units of m_{sun}. plotted are the semi major axes on the left column and the eccentricities of the three masses on the right column versus the resonance proximity parameter omega=0 +/-0.01

Figure 3: for 3 equal masses the change of e_1 is shown for different resonance parameters around w=0 for $10^{-5} < m_i < 10^{-3}$ with growing masses (3, 5,7 and 9 times the basic mass unit. i_4 decreases for larger values of e_1: $0.00 < e_1< 0.02$; but for e_2 the range is $0.00 < e_2 < 0.15$ and for e_3 the range is $0.00 < e_3 < 0.03$. Generally the eccentricies increase with w.

Figure 5: Influence of the variation of the middle mass m_2 on a_k (left column) and e_k (right column) and i_4 for $m_2/10^{-5}$ in[1,9]and $m_1=m_3=10^{-5}$