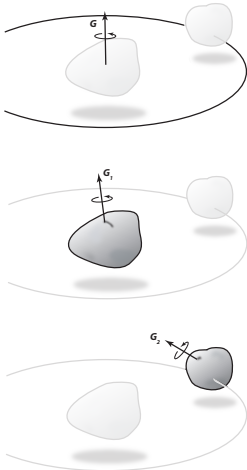


Solid tides: The Newtonian creep model and alternative rheologies

Gwenaél Boué, Alexandre Correia, Jacques Laskar & Clodoaldo Ragazzo



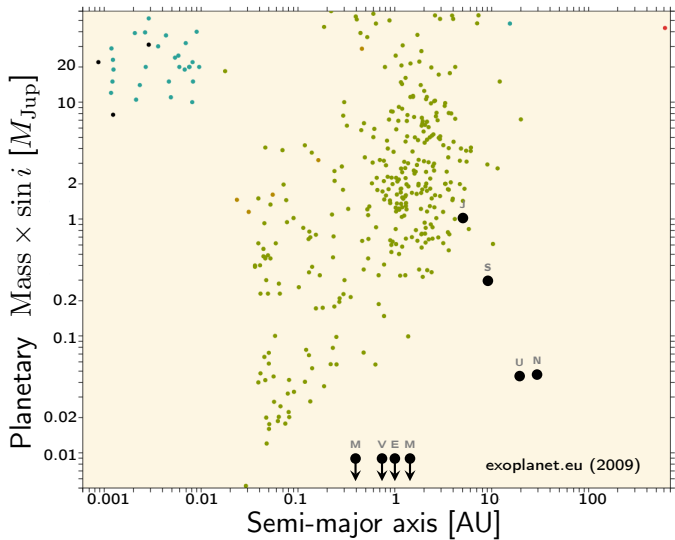
GB: Spin axis evolution of two interaction bodies



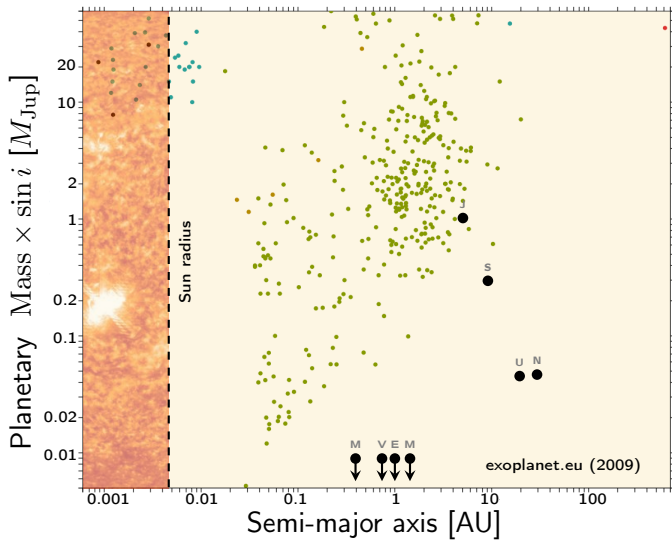
SFM: Friction in close-in satellites and exoplanets



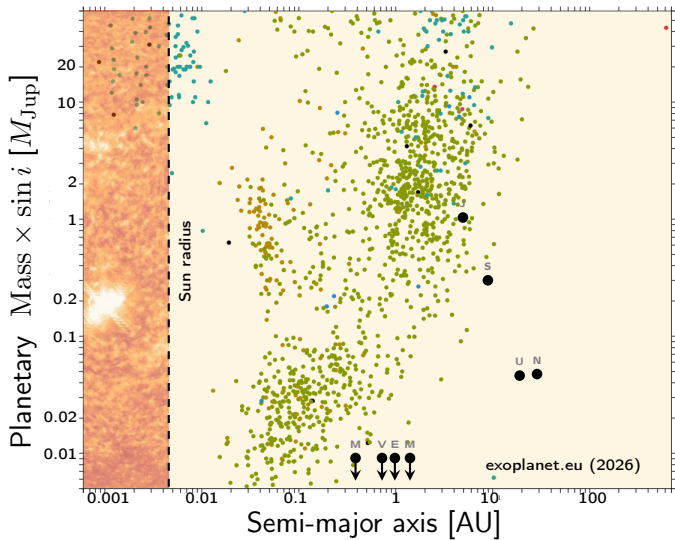
Exoplanets



Exoplanets



Exoplanets



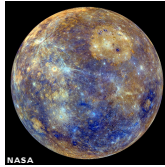
Tides in the solar system



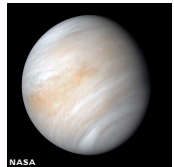
Earth



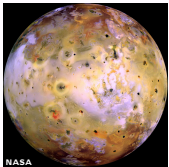
Moon



Mercury



Venus



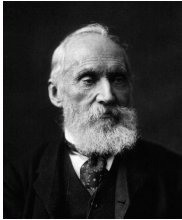
Io



Galilean moons

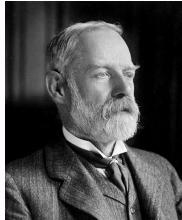
And many
more . . .

Tidal theory



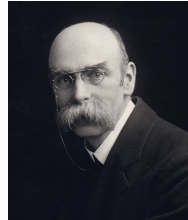
W. Thomson 1863

Elastic
deformation



G. Darwin 1878

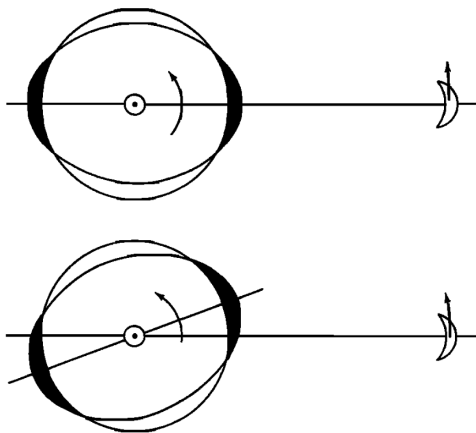
Viscous
deformation



A. Love 1909

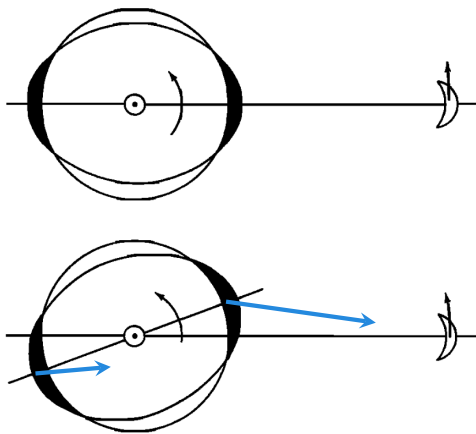
Love
numbers

Tidal mechanism



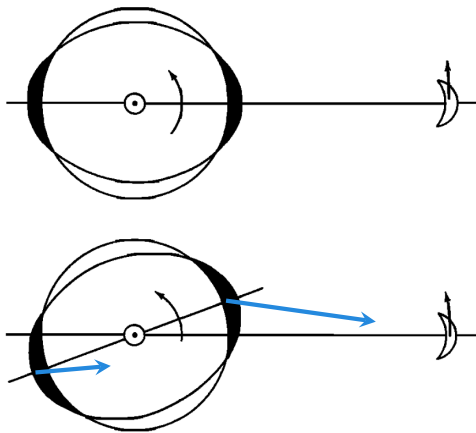
MacDonald 1964

Tidal mechanism



MacDonald 1964

Tidal mechanism

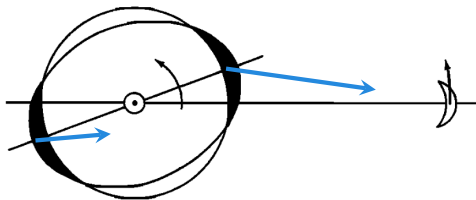
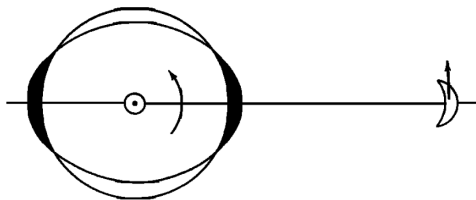


MacDonald 1964

Tidal effects

- Angular momentum exchange
- Energy dissipation

Tidal mechanism



MacDonald 1964

Tidal effects

- Angular momentum exchange
- Energy dissipation

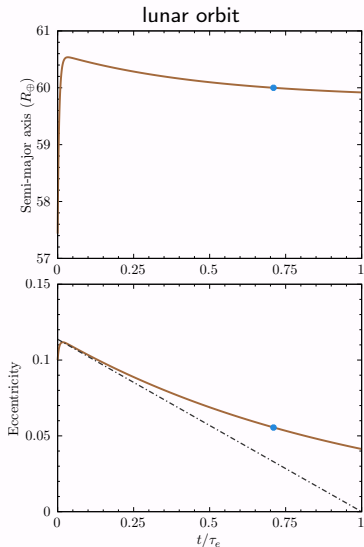
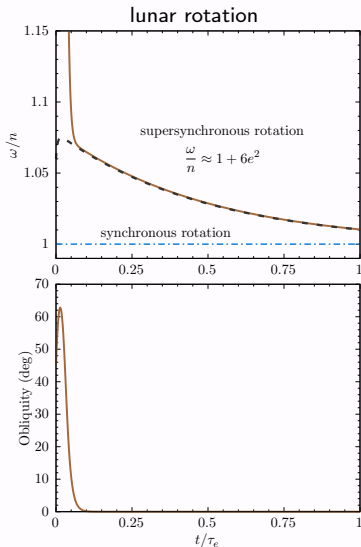
Simple prescriptions

- Fixed phase lag δ
(MacDonald 1964)
- Fixed time lag Δt
(Singer 1968)

Earth-moon evolution

assumption: tides in the moon only

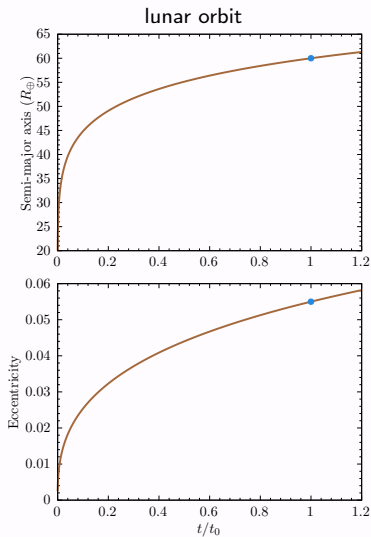
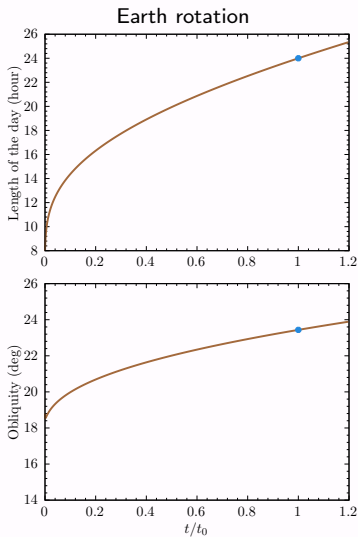
prescription: constant time lag model



Earth-moon evolution

assumption: tides in the Earth only

prescription: constant time lag model



Constant time lag model

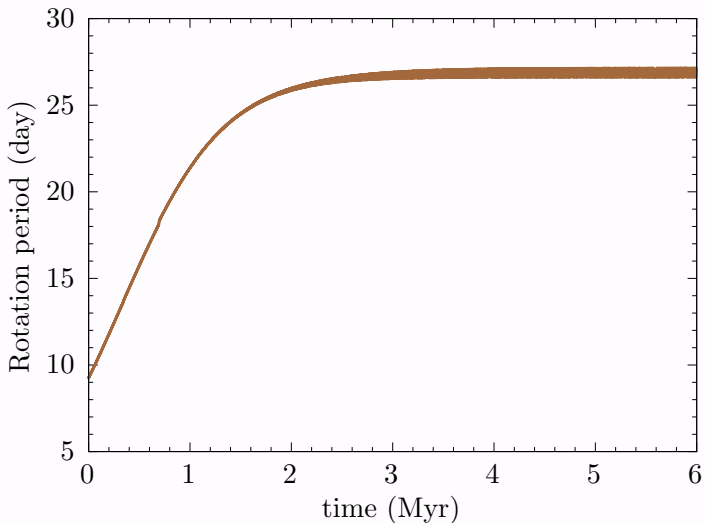
Main characteristics :

- Exchange of angular momentum ✓
- Energy dissipation ✓
- Pseudosynchronous rotation ✗

Moon's rotational evolution

assumption: tides in the moon only

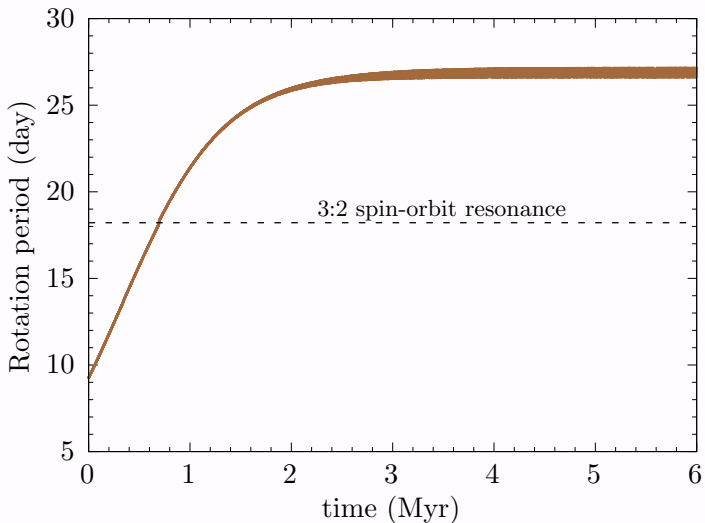
prescription: constant time lag model + permanent triaxiality



Moon's rotational evolution

assumption: tides in the moon only

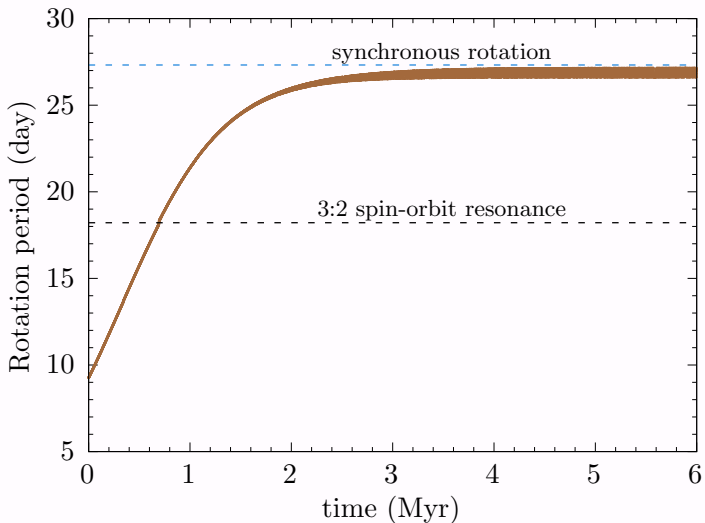
prescription: constant time lag model + permanent triaxiality



Moon's rotational evolution

assumption: tides in the moon only

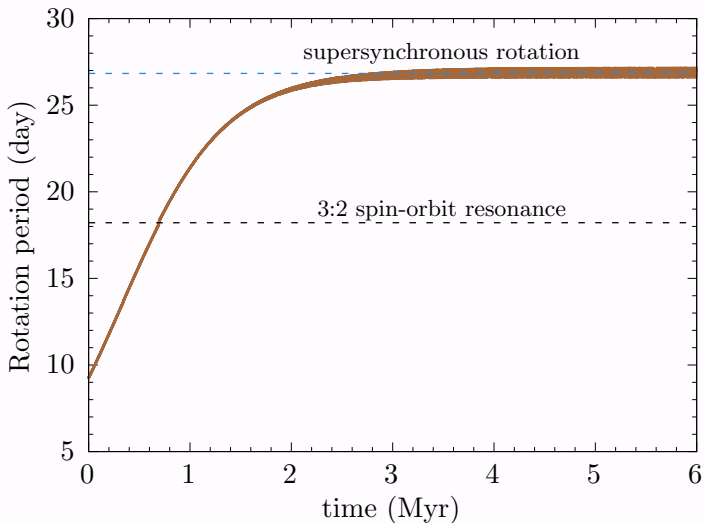
prescription: constant time lag model + permanent triaxiality



Moon's rotational evolution

assumption: tides in the moon only

prescription: constant time lag model + permanent triaxiality



Total torque

$$\ddot{\theta} = -\frac{6Gm_0mR^2}{Cr^3} \left[\underbrace{\frac{k_2\Delta t}{2} \frac{m_0}{m} \left(\frac{R}{r}\right)^3 (\dot{\theta} - \dot{f})}_{\text{tides}} + \underbrace{C_{22} \sin(2\theta - 2f)}_{\text{triaxiality}} \right]$$

Total torque

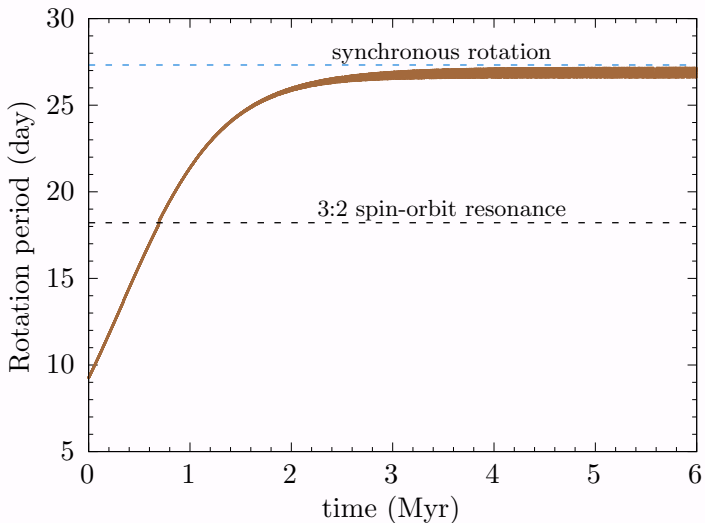
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At supersynchronous rotation :
Both terms average to zero !

Moon's rotational evolution

assumption: tides in the moon only

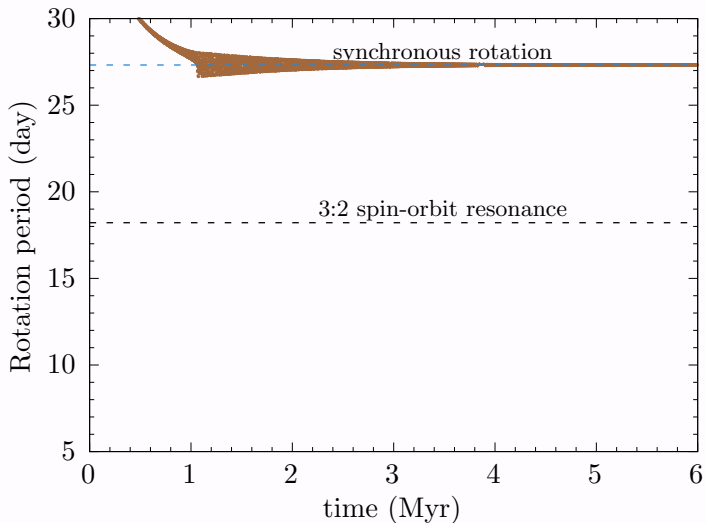
prescription: constant time lag model + permanent triaxiality



Moon's rotational evolution

assumption: tides in the moon only

prescription: constant time lag model + permanent triaxiality



From the CTL model to the creep tide model

Love number and transfer function

At the mean radius R :

- Generating potential : $U_T(\mathbf{R}, t)$
- Deformation potential : $U_D(\mathbf{R}, t)$

$$U_D(\mathbf{R}, t) = k_2 U_T(\mathbf{R}, t - \Delta t)$$

(Love 1909) + (Singer 1968)

Love number and transfer function

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(Love 1909) + (Singer 1968)

↓

$$\begin{aligned} U_D(\mathbf{R}, t) &= k_2(t) * U_T(\mathbf{R}, t) \\ &= \int_{-\infty}^{\infty} k_2(t - t') U_T(\mathbf{R}, t') dt' \end{aligned}$$

Transfer function

Love number and transfer function

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- Generating potential : $U_T(\mathbf{R}, t)$
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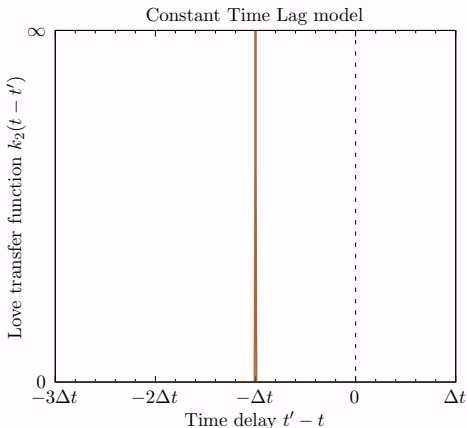
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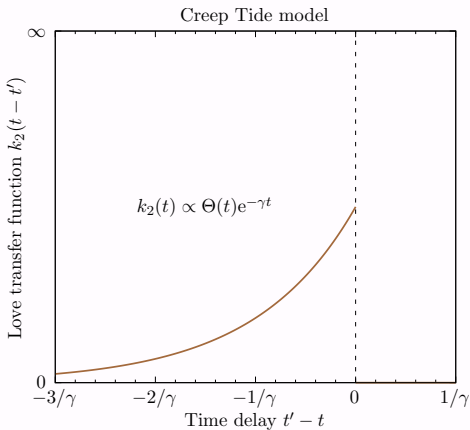
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Transfer function



Creep tide model

- Differential equation governing the tidal response :

$$\dot{U}_D(\mathbf{R}, t) + \gamma U_D(\mathbf{R}, t) = \gamma k_f U_T(\mathbf{R}, t)$$

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- Fourier transform :

$$i\omega \hat{U}_D(\mathbf{R}, \omega) + \gamma \hat{U}_D(\mathbf{R}, \omega) = \gamma k_f \hat{U}_T(\mathbf{R}, \omega)$$

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$$\hat{U}_D(\mathbf{R}, \omega) = \hat{k}_2(\omega) \hat{U}_T(\mathbf{R}, \omega), \quad \hat{k}_2(\omega) = \frac{k_f}{1 + i\omega/\gamma}.$$

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- Low frequency regime :

$$U_D(\mathbf{R}, t) = k_f \left(U_T(\mathbf{R}, t) - \frac{1}{\gamma} \dot{U}_T(\mathbf{R}, t) \right)$$

Creep tide model

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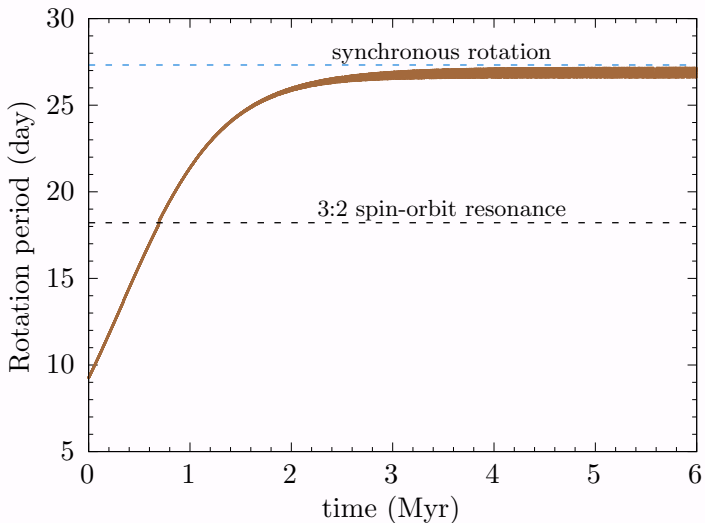
- Low frequency regime :

$$U_D(\mathbf{R}, t) = k_f \left(U_T(\mathbf{R}, t) - \frac{1}{\gamma} \dot{U}_T(\mathbf{R}, t) \right) \quad (\text{CTL model : } \frac{1}{\gamma} \leftrightarrow \Delta t)$$

Moon's rotational evolution

assumption: tides in the moon only

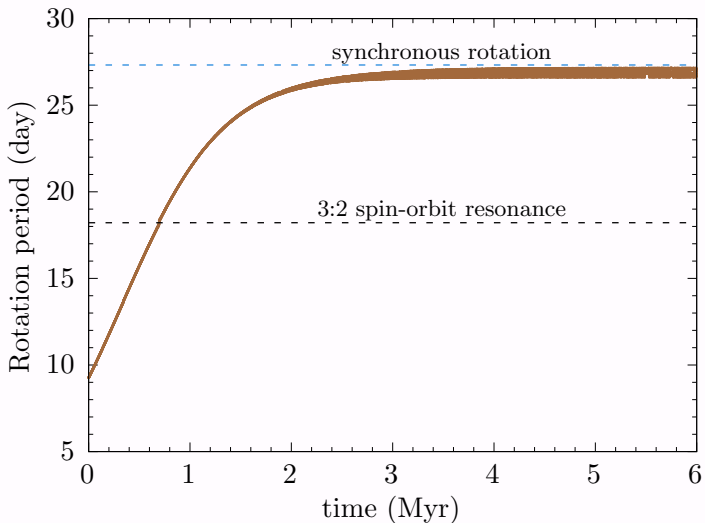
prescription: constant time lag model + permanent triaxiality



Moon's rotational evolution

assumption: tides in the moon only

prescription: creep tide model + permanent triaxiality



Choosing the right parameters...

Viscous timescales

- Time delays in INPOP19a :

$$\Delta t_{\text{Earth}} \approx 4.1 \text{ min}$$

$$\Delta t_{\text{Moon}} \approx 2.3 \text{ hr}$$

Viscous timescales

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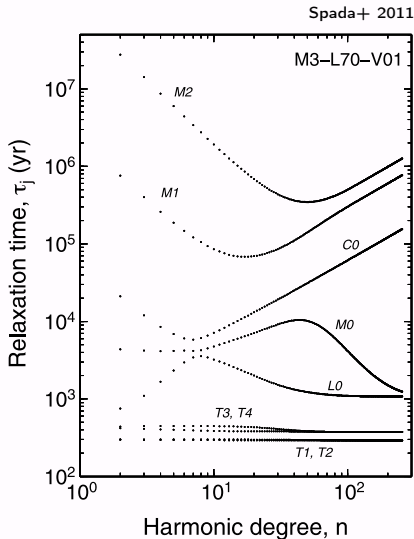
$$\Delta t_{\text{Earth}} \approx 4.1 \text{ min}$$

$$\Delta t_{\text{Moon}} \approx 2.3 \text{ hr}$$

- Earth viscous times

between 300 yr and 30 Myr

(postglacial rebound since $\approx 20,000$ yr)



Tidal torque

- **Expression** (Correia+ 2014) :

$$\Gamma_{\text{tide}} = -24 \frac{Gm^2}{R} \sum_k \text{Im} \left(\hat{C}_{22,T}(\omega_k) \hat{S}_{22,T}^*(\omega_k) \right) \text{Im} \left(\hat{k}_2(\omega_k) \right)$$

Tidal torque

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- **Outside spin-orbit resonance** :

$$\Gamma_{\text{tide}} \approx \frac{3}{2} \frac{Gm_0^2 R^5}{a^6} \text{Im} \left(\hat{k}_2(2\Omega - 2n) \right)$$

- **At synchronous rotation** :

$$\Gamma_{\text{tide}} \approx 18 \frac{Gm_0^2 R^5}{a^6} e^2 \text{Im} \left(\hat{k}_2(n) \right)$$

Tidal torque

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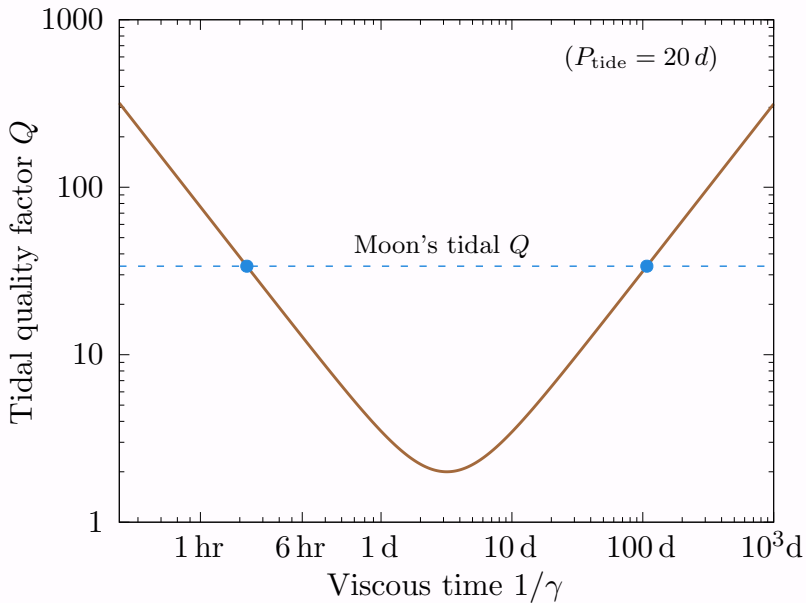
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- **Quality factor** :

$$\text{Im} \left(\hat{k}_2(\omega) \right) \equiv -\frac{k_2}{Q}$$

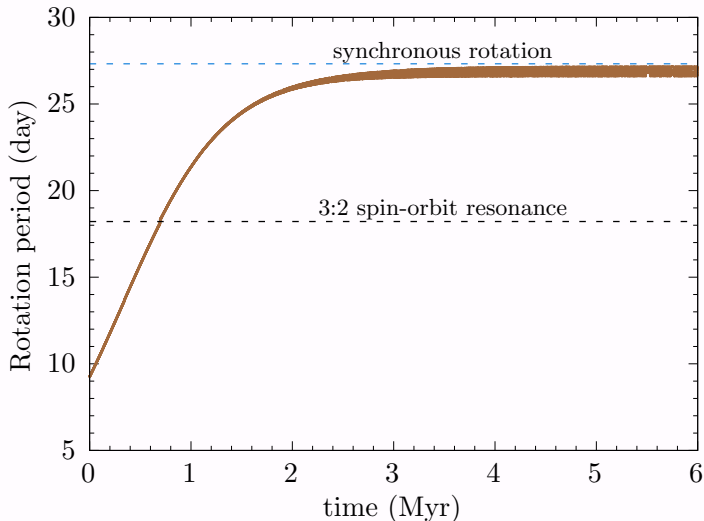
Tidal quality factor of the creep tide model



Moon's rotational evolution

assumption: tides in the moon only

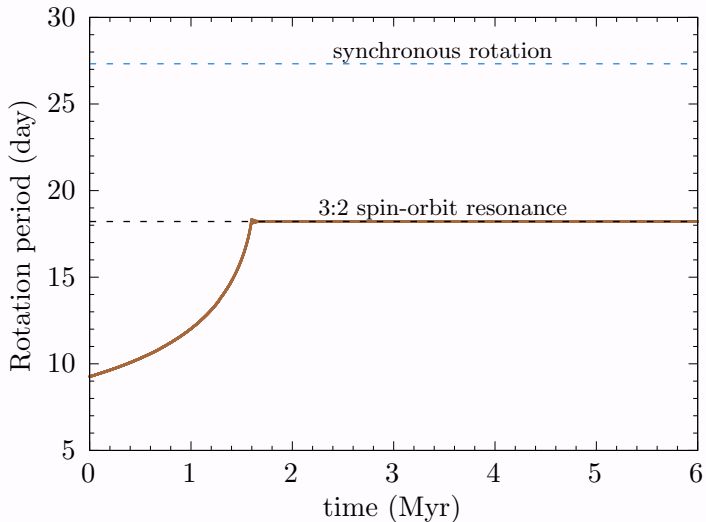
prescription: creep tide model (first branch) + permanent triaxiality



Moon's rotational evolution

assumption: tides in the moon only

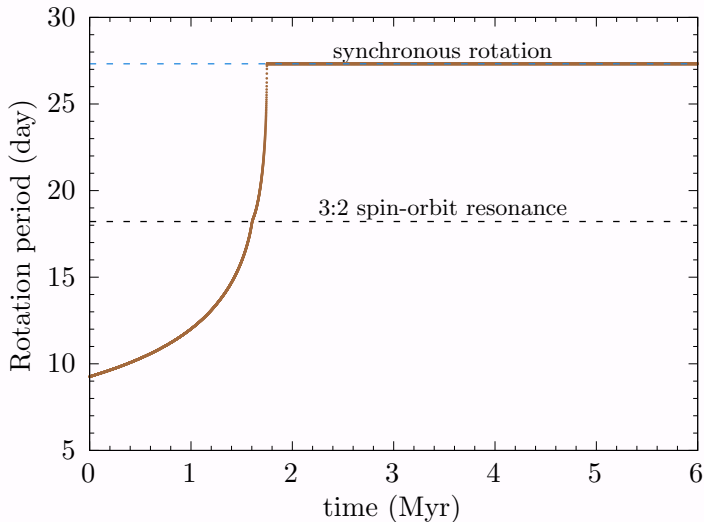
prescription: creep tide model (second branch) + permanent triaxiality



Moon's rotational evolution

assumption: tides in the moon only

prescription: creep tide model (second branch) without permanent triaxiality



On the spin-orbit resonances

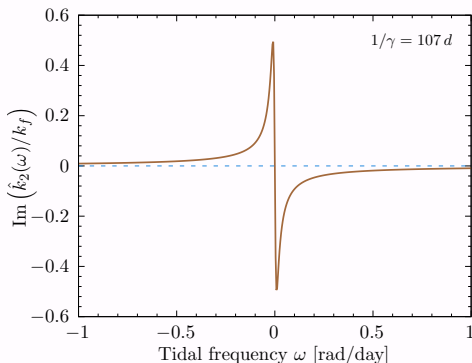
Creep tidal torque

- **Expression** (Correia+ 2014) :

$$\Gamma_{\text{tide}} = -24 \frac{Gm^2}{R} \sum_k \text{Im} \left(\hat{C}_{22,T}(\omega_k) \hat{S}_{22,T}^*(\omega_k) \right) \text{Im} \left(\hat{k}_2(2\Omega - kn) \right)$$

- **Imaginary part of the Love number** (Makarov & Efroimsky 2013):

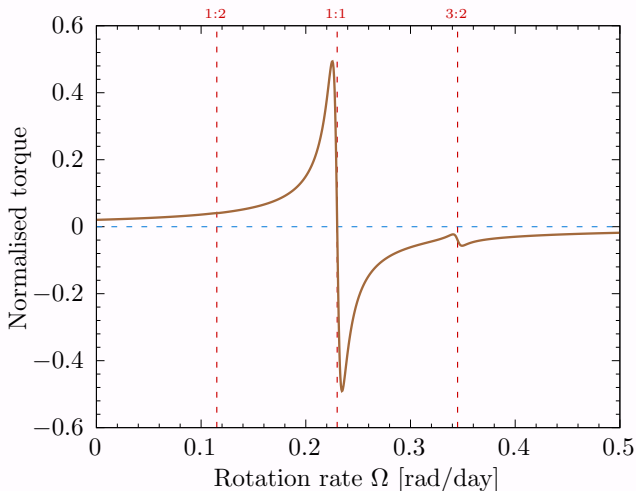
$$\text{Im} \left(\hat{k}_2(\omega) \right) = -k_f \frac{\omega/\gamma}{1 + (\omega/\gamma)^2}$$



Creep tidal torque

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Equilibrium rotation rate

- Constant time lag model :

$$\frac{\Omega}{n} = 1 + 6e^2 + \mathcal{O}(e^4)$$

Equilibrium rotation rate

- Constant time lag model :

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- Creep tide model (Ferraz-Mello 2013) :

$$\frac{\Omega}{n} = 1 + \frac{6e^2}{1 + n^2/\gamma^2} + \mathcal{O}(e^4)$$

Equilibrium rotation rate

- Constant time lag model :

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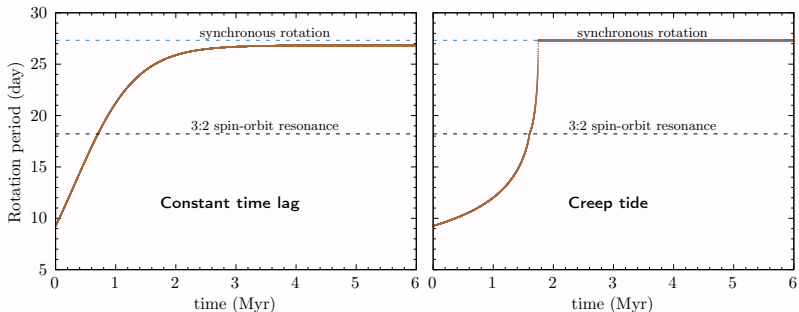
- Creep tide model (Ferraz-Mello 2013) :

$$\frac{\Omega}{n} = 1 + \frac{6e^2}{1 + n^2/\gamma^2} + \mathcal{O}(e^4)$$

- Arbitrary tidal model :

$$\frac{\Omega}{n} = 1 + 6e^2 \frac{\hat{k}_{2,i}(n)}{n \hat{k}'_{2,i}(0)} + \mathcal{O}(e^4), \quad \hat{k}_{2,i} \equiv \text{Im} \left(\hat{k}_2 \right)$$

Effect of the rheology on the long-term evolution



Tidal response for arbitrary rheology

Equations of motion

- **Constant time lag** (Correia+ 2011) :

$$\dot{a} = \frac{2K}{\beta a^2} \left(f_2(e) \frac{\omega}{n} - f_3(e) \right) a$$

$$\dot{e} = \frac{9K}{\beta a^2} \left(\frac{11}{18} f_4(e) \frac{\omega}{n} - f_5(e) \right) e$$

$$\dot{\Omega} = -\frac{Kn}{C} \left(f_1(e) \frac{\omega}{n} - f_2(e) \right)$$

(Secular equations)

Equations of motion

- **Creep tidal model** (Correia+ 2014) :

$$\ddot{\mathbf{r}} = -\frac{\mu_0}{r^2} \hat{\mathbf{r}} - \frac{3\mu_0 R^2}{2r^4} J_2 \hat{\mathbf{r}} - \frac{9\mu_0 R^2}{r^4} \left[C_{22} \cos 2\delta - S_{22} \sin 2\delta \right] \hat{\mathbf{r}} \\ + \frac{6\mu_0 R^2}{r^4} \left[C_{22} \sin 2\delta + S_{22} \cos 2\delta \right] \mathbf{K} \times \hat{\mathbf{r}}$$

$$\ddot{\theta} = -\frac{6Gm_0 m R^2}{Cr^3} \left[C_{22} \sin 2\delta + S_{22} \cos 2\delta \right]$$

$$\dot{J}_2 + \gamma J_2 = \gamma k_f J_2^T$$

$$\dot{C}_{22} + \gamma C_{22} = \gamma k_f C_{22}^T$$

$$\dot{S}_{22} + \gamma S_{22} = \gamma k_f S_{22}^T$$

(Instantaneous equations in the time domain)

Equations of motion

- **Arbitrary tidal model** (Kaula 1964) :

$$\dot{a} = -\frac{1}{2} \frac{m_0}{m} \left(\frac{R}{a}\right)^5 \sum_{p=-\infty}^{\infty} \left[(X_p^{-3,0})^2 pn \hat{k}_{2,i}(pn) - 3 (X_p^{-3,2})^2 pn \hat{k}_{2,i}(\omega_p) \right] a$$

$$\dot{e} = \frac{1-e^2}{e} \left[\frac{C\dot{\Omega}}{\beta n a^2 \sqrt{1-e^2}} + \frac{\dot{a}}{2a} \right]$$

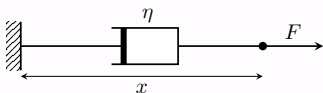
$$\dot{\Omega} = \frac{3}{2} \frac{\beta R^2}{C} \frac{m_0}{m} \left(\frac{R}{a}\right)^3 \left[\sum_{p=-\infty}^{\infty} (X_p^{-3,2})^2 \hat{k}_{2,i}(\omega_p) \right] n^2 \quad (\omega_p = 2\Omega - pn)$$

(Secular equations in the Fourier domain)

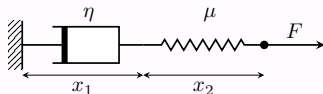
A few rheologies

models (e.g., Henning+ 2009) :

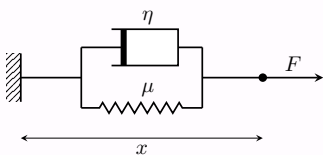
Newton creep rheology



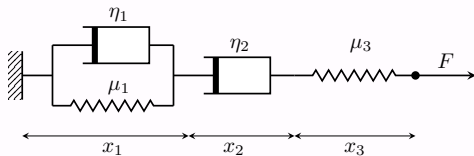
Maxwell rheology



Kelvin-Voigt rheology



Burger rheology



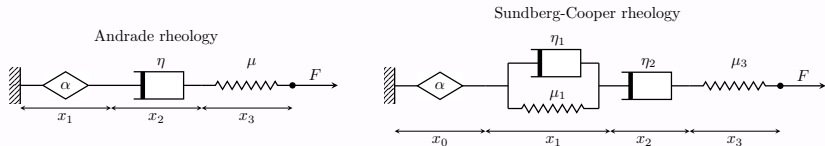
notation :

x : strain

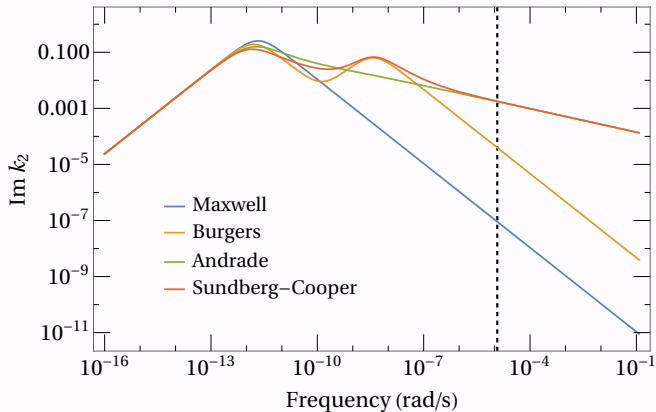
F : stress

$\mu_{\text{eff}} = \frac{F}{x}$: effective rigidity

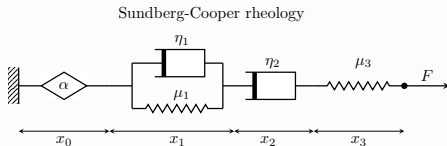
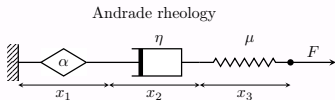
Andrade model



Model of TRAPPIST-1e Love number (Gevorgyan 2021) :



Andrade model



Andrade complex rigidity :

$$\hat{\mu}_{\text{eff}}(\omega) = \frac{\mu_0}{1 + \Gamma(1 + \alpha)(i\omega\tau_A)^{-\alpha} + (i\omega\tau_M)^{-1}}$$

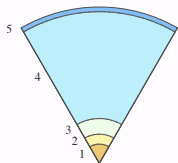
Fractional derivative $\frac{d^\alpha}{dt^\alpha}$??

Extended Burger model (Gevorgyan 2026)

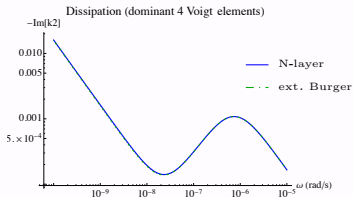
Interior model :

Love number :

Layered body — cross-section (visual thickness exaggerated)

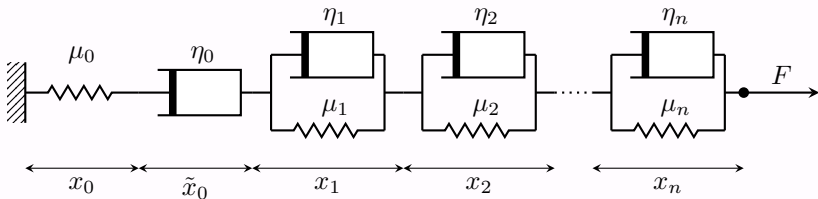


- 1 — Inner core 213.5 km (213.5 km)
- 2 — Outer core 111.5 km (325.0 km)
- 3 — Lower mantle 175.0 km (500.0 km)
- 4 — Upper mantle 1197.2 km (1697.2 km)
- 5 — Crust 40.0 km (1737.2 km)



Approximated rheology :

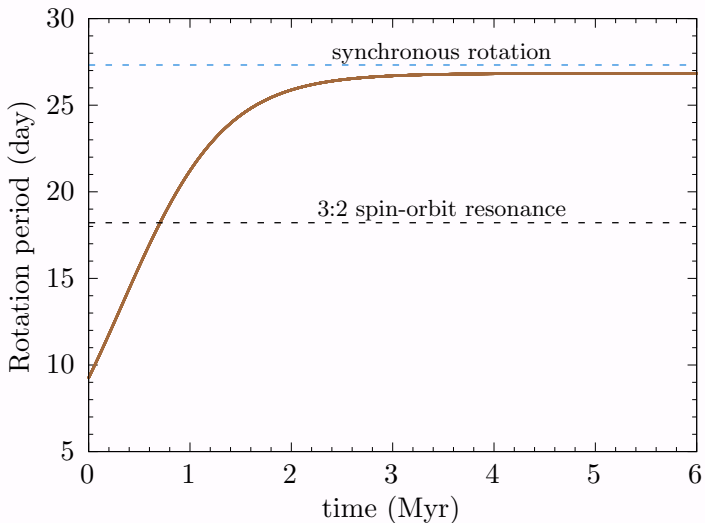
extended Burger rheology



Moon's rotational evolution

assumption: tides in the moon only

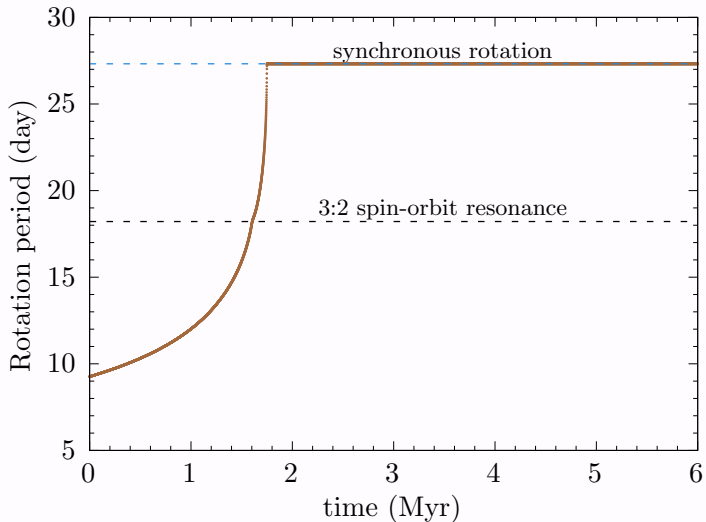
prescription: constant time lag model



Moon's rotational evolution

assumption: tides in the moon only

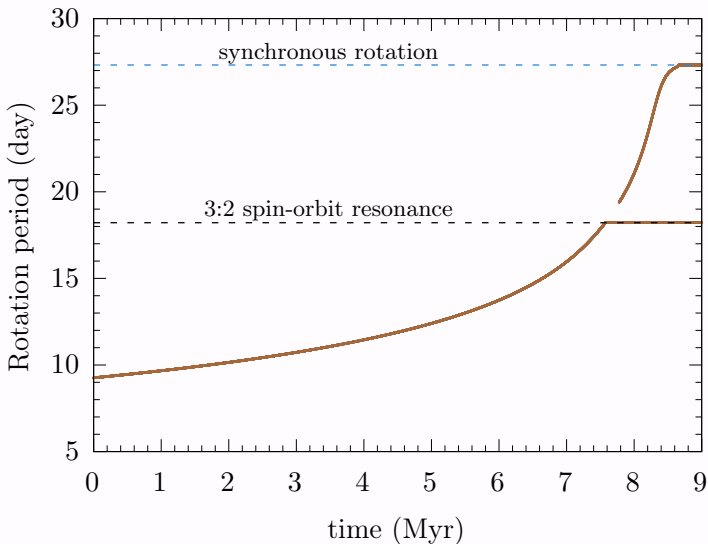
prescription: creep tide model (second branch)



Moon's rotational evolution

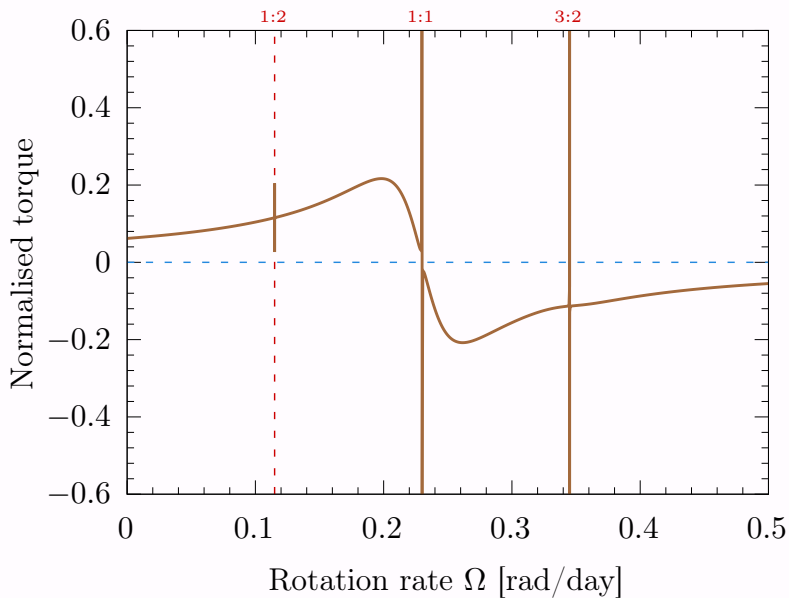
assumption: tides in the moon only

prescription: Layered model



Thank you !

Normalised tidal torque of the layered model



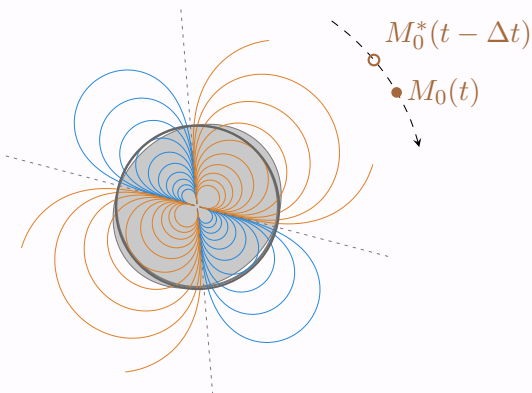
Love number

At the planet surface :

- Generating potential : $U_T(\mathbf{R}, t)$
- Potential due to the mass redistribution : $U_D(\mathbf{R}, t)$

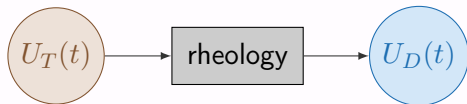
$$U_D(\mathbf{R}, t) = k_2 U_T(\mathbf{R}, t - \Delta t)$$

(Love 1909) + (Singer 1968)



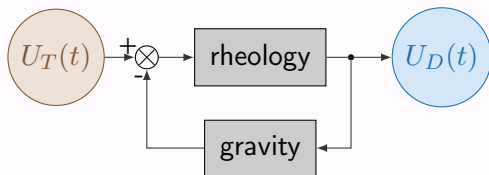
Homogeneous elastic body

Transfer function :



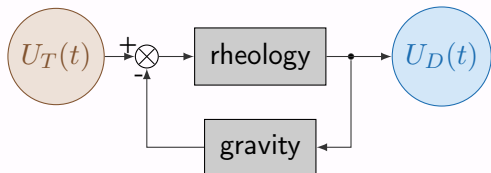
Homogeneous elastic body

Transfer function :



Homogeneous elastic body

Transfer function :



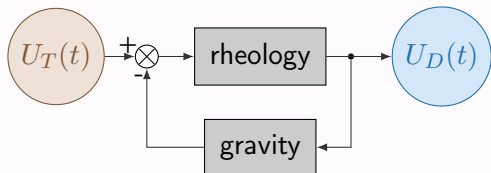
Love number :

$$k_2 = \frac{3}{2} \frac{1}{1 + \frac{25}{2} \frac{\mu_{\text{eff}}}{\rho g R}}$$

(approximation: Ragazzo & Ruiz 2015)

Homogeneous elastic body

Transfer function :



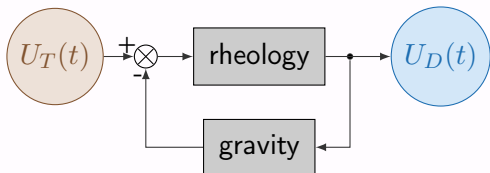
Love number :

$$k_2 = \frac{3}{2} \frac{1}{1 + \frac{4}{2} \frac{\mu_{\text{eff}}}{\rho g R}}$$

(approximation: Ferraz-Mello 2013)

Homogeneous elastic body

Transfer function :



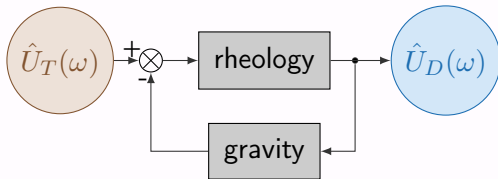
Love number :

$$k_2 = \frac{3}{2} \frac{1}{1 + \frac{19}{2} \frac{\mu_{\text{eff}}}{\rho g R}}$$

(exact calculation: Thomson 1863)

Correspondence principle

Transfer function :



Love number :

$$\hat{k}_2(\omega) = \frac{3}{2} \frac{1}{1 + \frac{19}{2} \frac{\hat{\mu}_{\text{eff}}(\omega)}{\rho g R}}$$

(exact calculation: Darwin 1879)

Tidal differential equation

In the Fourier domain :

$$\hat{C}_{22}(\omega) = \hat{k}_2(\omega) \hat{C}_{22}^T(\omega) \quad (\text{idem } J_2, S_{22}, \dots)$$

Love number :

$$\hat{k}_2(\omega) = \frac{a_0 + a_1(i\omega) + a_2(i\omega)^2 + \dots}{b_0 + b_1(i\omega) + b_2(i\omega)^2 + \dots}$$

Time domain :

$$b_0 C_{22} + b_1 \frac{dC_{22}}{dt} + b_2 \frac{d^2 C_{22}}{dt^2} + \dots = a_0 C_{22}^T + a_1 \frac{dC_{22}^T}{dt} + a_2 \frac{d^2 C_{22}^T}{dt^2} + \dots$$