Proper elements for space debris

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Outline

1. Space debris

2. Proper elements and space debris

3. Machine Learning and space debris clustering

4. Dissipative proper elements

Satellites and space debris

• Sputnik 1 was the first artificial satellite, launched by the Soviet Union on 4 October 1957.

▷ Number of rocket launches since the start of the space age in 1957
About 7170 (excluding failures)

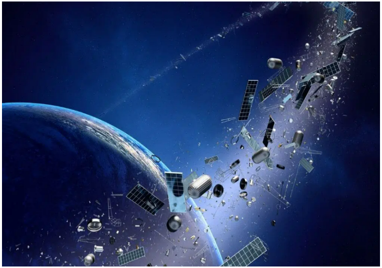
▷ Number of satellites these rocket launches have placed into Earth orbit
About 25170

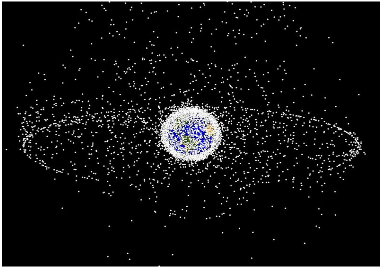
▷ Number of these still in space
About 16910

▷ Number of these still functioning
About 14200

▷ ▷ ▷ Several millions of space debris.

Space debris





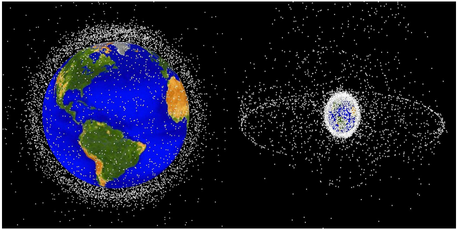
- Space debris:

▷ 54000 > 10 cm

▷ 1.2 millions between 1-10 cm

▷ 140 millions between 1 mm - 1 cm.

Space debris



• A collision with a:
– 10 cm object could entail a catastrophic fragmentation of a typical satellite;
– 1 cm object could disable a spacecraft and penetrate the ISS shields;
– 1 mm object could destroy sub-systems on board a spacecraft.

• In a historical overview in 2009, the NASA scientist Donald J. Kessler stated: "*Aggressive space activities without adequate safeguards could significantly shorten the time between collisions and produce an intolerable hazard to future spacecraft.*"

Dynamics of satellites and space debris

Why to study the dynamics of satellites and space debris?

1. Dynamics is essential for maintenance, control and mitigation strategies.
2. Possible end-of-life disposal strategies: graveyard orbits.

All aspects of the dynamics are useful:

1. stable regions, to avoid interactions between satellites and debris;
2. unstable regions, to move to graveyard orbits or re-enter in atmosphere.

Difficulties:

1. dissipative system in LEO, conservative system in MEO and GEO;
2. different time-scales: fast, semi-fast, slow angles.

(Recent) authors: Alessi, Colombo, De Blasi, Dogkas, Efthymiopoulos, Gachet, Gales, Gkolias, Howell, Lemaitre, Pucacco, Rosengren, Rossi, Sampaio, Tsiganis, Valsecchi, Vartolomei, Vilhena de Moraes, etc.

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3. Machine Learning and space debris clustering

4. Dissipative proper elements

Proper elements

• Under a non-resonance condition on the frequency and assuming a trigonometric perturbation, transform

$$\mathcal{H}(\underline{J}, \underline{\varphi}) = Z(\underline{J}) + \varepsilon R(\underline{J}, \underline{\varphi})$$

into

$$\mathcal{H}_K(\underline{J}', \underline{\varphi}') \ = \ Z_K(\underline{J}') + \varepsilon^K R_K(\underline{J}', \underline{\varphi}') \ .$$

• *Proper elements* are true integrals of the simplified Hamiltonian Z_K (since $\dot{\underline{J}}' = \underline{0}$) and quasi-integrals for the full Hamiltonian (since $\dot{\underline{J}}' = -\varepsilon^K \frac{\partial R_K(\underline{J}', \underline{\varphi}')}{\partial \underline{\varphi}'}$); they act as fingerprints of the dynamics, and can be used to classify objects into families.

• Proper elements stay nearly constant. Fixing an excursion Δ_J:

$$|\underline{J}'(T) - \underline{J}'(0)| \leq \varepsilon^K \| \frac{\partial R_K(\underline{J}', \underline{\varphi}')}{\partial \underline{\varphi}'} \| \, T \ = \ \Delta_J \ ,$$

which gives a stability time:

$$T = \frac{\Delta_J}{\varepsilon^K \| \frac{\partial R_K(\underline{J}', \underline{\varphi}')}{\partial \underline{\varphi}'} \|} \ .$$

Proper elements for asteroid families

▷ Hirayama (1918): used proper elements leading to asteroid families

▷ Brouwer (1951): with an improved theory of planetary motion

▷ Williams (1969): semianalytic theory of asteroid secular perturbations

▷ Kozai (1979): high-inclination asteroids

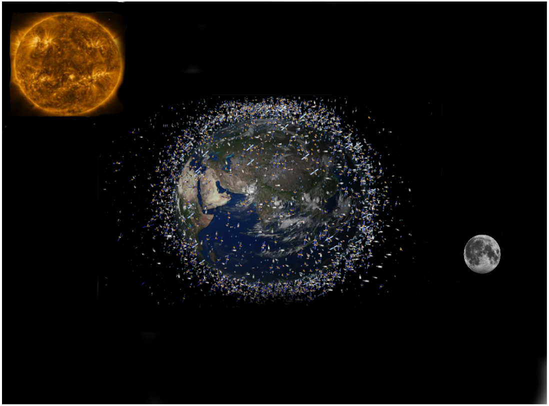
▷ Schubart (1982, 1991): proper parameters for resonant groups, i.e., for Hildas and Trojans

▷ Knezevic and Milani: analytical and synthetic proper elements (since 1988).

Model

Geolunisolar model: includes

1. Earth (with a non-spherical shape)
2. Sun
3. Moon (treated as third-body perturbations).



Hamiltonian model

• Action–angle Delaunay variables:

▷ Actions L, G, H, related to the orbital elements a, e, i by

$$L = \sqrt{\mathcal{G}\, m_E\, a}\ , \qquad G = L\sqrt{1-e^2}\ , \qquad H = G\cos i\ .$$

▷ Angles M, ω, Ω = mean anomaly, argument of perigee, longitude of the ascending node.

• Hamiltonian:

$$\mathcal{H} = -\frac{\mathcal{G}\, m_E}{2a} + \mathcal{H}_{Earth}(a,e,i,M,\omega,\Omega,\theta) + \mathcal{H}_{Moon}(a,e,i,M,\omega,\Omega,\Lambda_M)$$
$$+\mathcal{H}_{Sun}(a,e,i,M,\omega,\Omega,\Lambda_S) + \mathcal{H}_{SRP}(a,e,i,M,\omega,\Omega,\Lambda_S)$$

with θ = sidereal time, Λ_M, Λ_S = orbital elements of Moon and Sun.

Different timescales

• In the space debris problem, one has variables changing on different time scales, hence we will need a hierarchical perturbation theory to define the proper elements:

▷ fast angles: mean anomaly of the debris, sidereal time accounting for the rotation of the Earth (periods of days);

▷ semi–fast angles: mean anomalies of Moon and Sun (periods of 1 month and 1 year);

▷ slow angles: argument of perigee and longitude of the ascending node of debris, Moon and Sun (periods of years).

Osculating, mean, proper elements

Definition

Osculating orbital elements are obtained integrating the full Hamiltonian or Cartesian equations of motion.

Definition

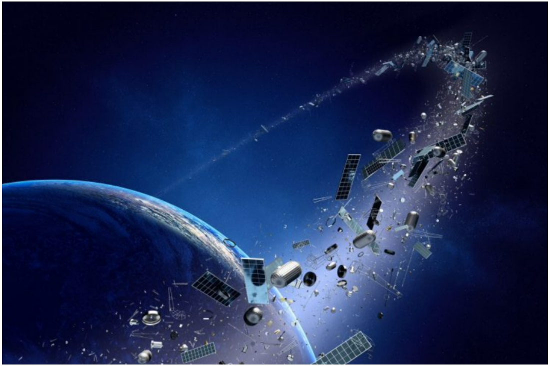
Mean orbital elements are the orbital elements obtained after averaging the Hamiltonian w.r.t. the short-period variables.

Definition

Proper elements are obtained averaging the Hamiltonian function w.r.t. fast, semi-fast and long-period variables; computed after implementing the normal form, they are quasi-integrals of motion.

Mathematical tools:

1. Perturbative methods: to compute approximate solutions.
2. Proper elements: which are quantities (nearly) constant over time.
3. Statistical techniques: to analyze the results.
4. Machine Learning: to cluster the fragments.



Mathematical tools:

1. Perturbative methods: to compute approximate solutions.
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Results

1. dynamics and stability of the fragments generated by a break-up event;
2. cluster space debris through proper elements;
3. reconnect the space debris to their parent bodies.

• Collaborations with Apetrii, De Blasi, Dogkas, Efthymiopoulos, Gachet, Gales, Lhotka, Pucacco, Vartolomei.

Dynamical systems tools

• In the neighborhood of a resonance, reduce the Hamiltonian to a pendulum-like one $\Rightarrow$ measure the amplitude of the resonances.

• Chaos might arise due to the increase of a parameter, which provokes a resonance overlap (Chirikov criterion).

• Analyze bifurcations for possible changes of stability of the equilibria or birth of periodic orbits.

• Fast Lyapunov Indicators discern numerically regular and chaotic motion.

• Use normal forms to reduce the Hamiltonian and give an estimate of the stability time.

• Develop exponential time stability estimates á la Nekhoroshev under suitable non-degeneracy conditions.

• Compute proper elements, quasi-integrals of motion to characterize the dynamics.

SIMPRO (NASA EVOLVE 4.0 - Johnson et al 2001)

Break-up event:

1. Explosion/Collision;
2. INPUT: orbital elements, minimum size of the generated fragments, mass of parent body and projectile, collision velocity, type of parent body
3. OUTPUT: information on each fragment, data analysis

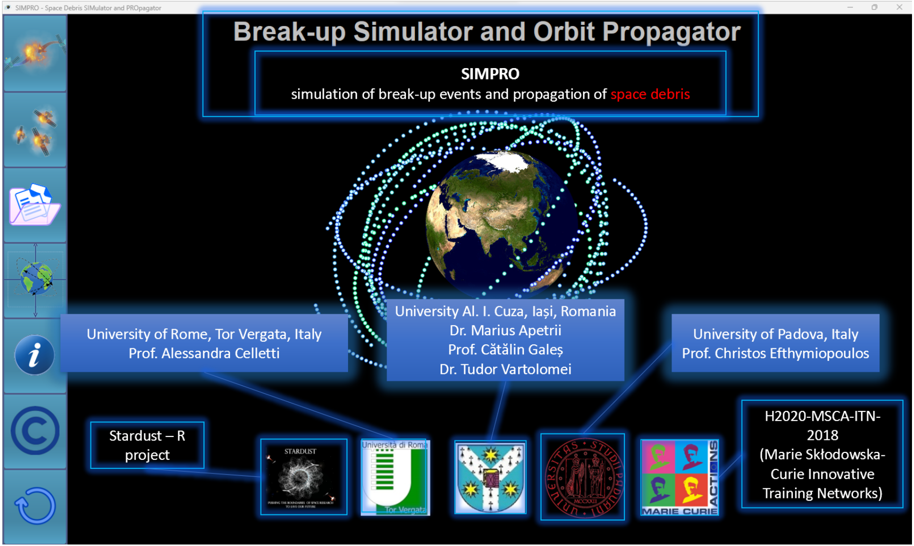
Propagator:

1. select the model;
2. select the equations of motion (Hamiltonian or Cartesian);
3. set the parameters (period and step);
4. OUTPUT: propagation of the orbital elements.

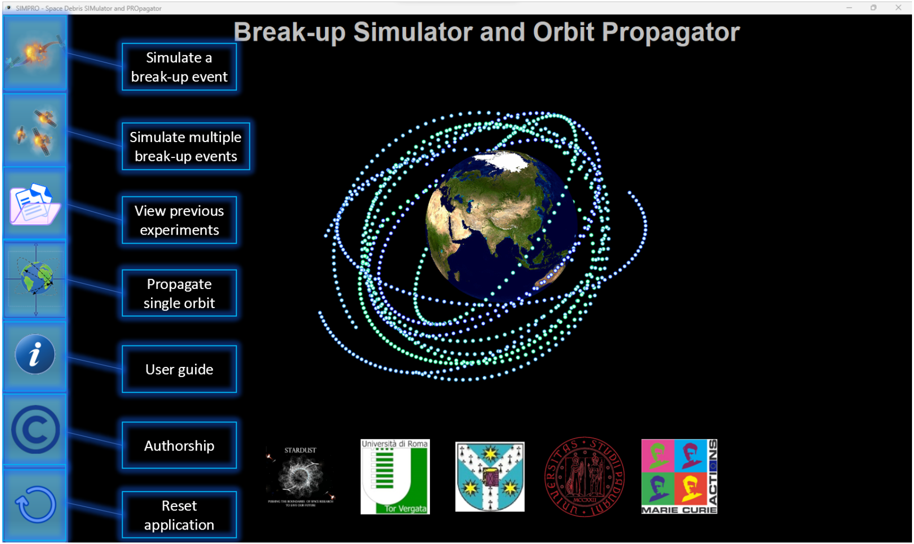
Executable SIMPRO is freely available on

https://github.com/simproproject/simpro_app

SIMPRO


SIMPRO - Space Debris SIMulator and PROpagator
Break-up Simulator and Orbit Propagator
SIMPRO
simulation of break-up events and propagation of space debris
University of Rome, Tor Vergata, Italy
Prof. Alessandra Celletti
University Al. I. Cuza, Iași, Romania
Dr. Marius Apetrii
Prof. Cătălin Galeș
Dr. Tudor Vartolomei
University of Padova, Italy
Prof. Christos Efthymiopoulos
Stardust – R
project
STARDUST
Università di Roma
Tor Vergata
MARIE CURIE
ACTIONS
H2020-MSCA-ITN-
2018
(Marie Skłodowska-
Curie Innovative
Training Networks)

SIMPRO


SIMPRO - Space Debris SIMulator and PROpagator
Break-up Simulator and Orbit Propagator
Simulate a break-up event
Simulate multiple break-up events
View previous experiments
Propagate single orbit
User guide
Authorship
Reset application
STARDUST
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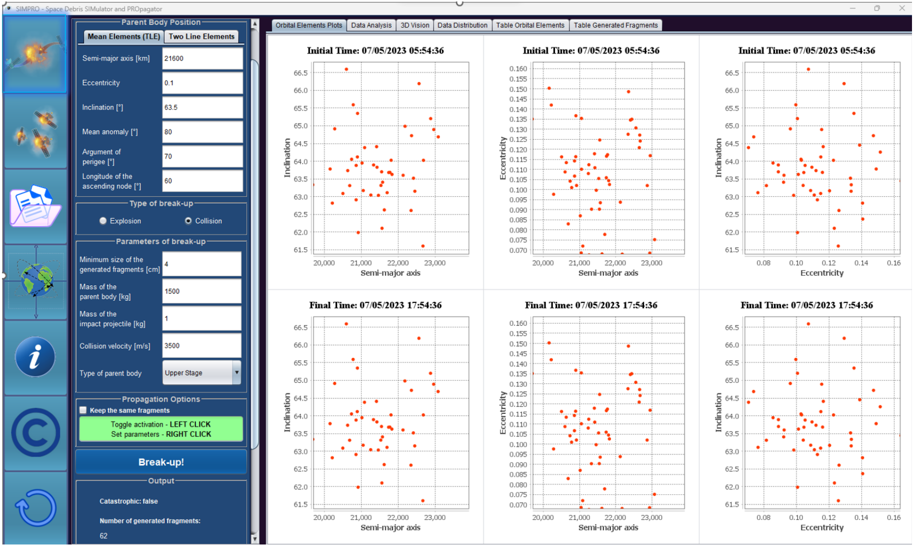
SIMPRO



SIMPRO - Space Debris SIMulator and PROpagator
Parent Body Position
Mean Elements (TLE)
Two Line Elements
Semi-major axis [km]
21600
Eccentricity
0.1
Inclination [°]
63.5
Mean anomaly [°]
80
Argument of perigee [°]
70
Longitude of the ascending node [°]
60
Type of break-up
Explosion
Collision
Parameters of break-up
Minimum size of the generated fragments [cm]
4
Mass of the parent body [kg]
1500
Mass of the impact projectile [kg]
1
Collision velocity [m/s]
3500
Type of parent body
Upper Stage
Propagation Options
Keep the same fragments
Toggle activation - LEFT CLICK
Set parameters - RIGHT CLICK
Break-up!
Output
Catastrophic: false
Number of generated fragments:
62
Orbital Elements Plots
Data Analysis
3D Vision
Data Distribution
Table Orbital Elements
Table Generated Fragments
Initial Time: 07/05/2023 05:54:36
Final Time: 07/05/2023 17:54:36
Inclination
Eccentricity
Semi-major axis

Outline

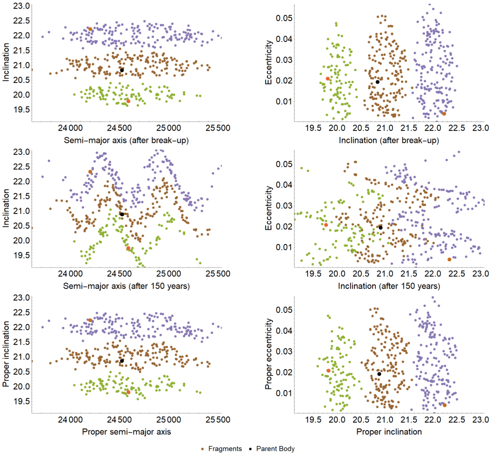
Proper elements: three groups clustering

• Proper elements are of fundamental importance for the clustering space debris, i.e. finding a map from datapoints to categories, based on the distribution of points in space.

proper/mean elements at the initial time

mean elements after 150 years

proper elements after 150 years



Phase space of $a - i$ (left), $e - i$ (right) at the initial position $a = 24600$ km, $e = 0.02$, $\omega = 110^o$, $\Omega = 120^o$ with $i = 20^o$ AND $i = 21^o$ AND $i = 22^o$.

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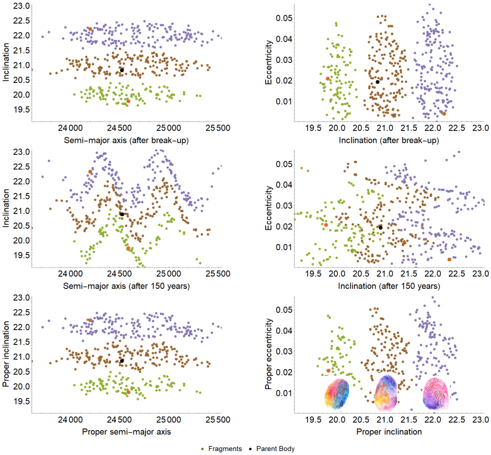
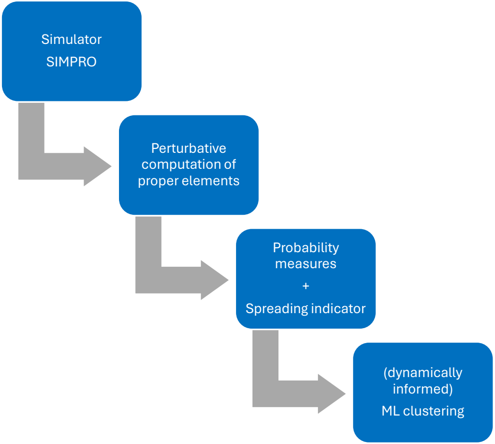
proper elements after 150 years



Phase space of $a - i$ (left), $e - i$ (right) at the initial position $a = 24600$ km, $e = 0.02$, $\omega = 110^o$, $\Omega = 120^o$ with $i = 20^o$ AND $i = 21^o$ AND $i = 22^o$.

Work-flow

• Celletti A., Vartolomei T., "A dynamics based procedure for clustering and classifying space debris", Scientific Reports (2025)



Clustering methods

✓ Choice of the phase space region.

✓ Choice of the distance.

✓ Choice of the recording times.

✓ Compute the spreading indicator.

Phase space region and Distance

- Choice of the phase space region:

$(ps1)$ Avoid tesseral resonances (between $\dot{M}$, $\dot{\theta}$), provoking changes of a;

$(ps2)$ Avoid Lunisolar secular resonances (between $\dot{\omega}$, $\dot{\Omega}$, $\dot{M}_M$, $\dot{M}_S$), provoking changes of e, i;

$(ps3)$ Avoid secular resonances, like those depending only on the inclination as the result of the following commensurability relation: $k_1\dot{\omega} + k_2\dot{\Omega} = 0$ (at $i = 90^o, 46.4^o, 69.0^o, 123.9^o$, etc.).

- Choice of the distance:

$(d1)$ Euclidean distance: $d_1^{(k,\ell)} = \sqrt{(a_k - a_\ell)^2 + (e_k - e_\ell)^2 + (\sin i_k - \sin i_\ell)^2}$

$(d2)$ Perigee distance:
$d_2^{(k,\ell)} = \sqrt{(e_k - e_\ell)^2 + (\sin i_k - \sin i_\ell)^2 + ((1 - e_k)a_k - (1 - e_\ell)a_\ell)^2}$

$(d3)$ Canberra distance: $d_3^{(k,\ell)} = \frac{|a_k - a_\ell|}{|a_k| + |a_\ell|} + \frac{|e_k - e_\ell|}{|e_k| + |e_\ell|} + \frac{|\sin i_k - \sin i_\ell|}{|\sin i_k| + |\sin i_\ell|}$

$(d4)$ Bray-Curtis distance: $d_4^{(k,\ell)} = \frac{|a_k - a_\ell| + |e_k - e_\ell| + |\sin i_k - \sin i_\ell|}{|a_k + a_\ell| + |e_k + e_\ell| + |\sin i_k + \sin i_\ell|}$

Recording times and Spreading indicator

• Recording times: determine the elements of each fragment of all groups at

$(T1)$ fixed step size for all times in an interval (t_0, t_f);

$(T2)$ random times between (t_0, t_f);

$(T3)$ controlled shuffled times: control the shuffling by selecting the recording times of the fragments as values of a χ^2 distribution with 2 degrees of freedom.

• Spreading indicator, involving only e and i, since a is nearly constant (after averaging over the mean anomaly):

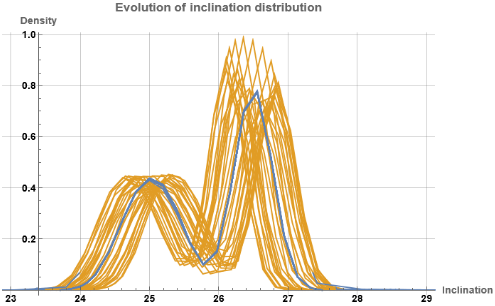
$$S = \frac{1}{2N_{frag}} \frac{1}{\sum_{k=2}^{N_t} \frac{1}{k}} \sum_{j=1}^{N_{frag}} \sum_{k=2}^{N_t} \Big(\frac{|e_k^{(j)} - e_1^{(j)}|}{k} + |\frac{|\sin i_k^{(j)}| - |\sin i_1^{(j)}|}{k}| \Big) .$$

Large S denotes big excursion in e and/or i; used to measure the spread of the elements and gives information on the applicability of the clusterization methods.

Check the variation of the elements

• Evolution of the probability density functions (PDF) of the mean and proper inclinations over 30 years.

• PDFs of mean inclinations vary, while proper inclination almost perfectly overlap, keeping the same shape over time and resulting from a mixture of 2 normal distributions.

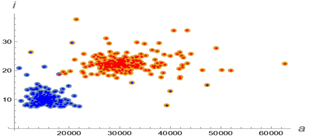


• The stability of the PDFs of the proper inclinations (or eccentricities) allows us to proceed to cluster the data.

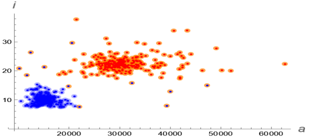
Select the clustering method

- Clustering methods: `KMeans`, `DBSCAN` (Density-based spatial clustering of applications with noise), `GaussianMixture` (models the probability density with a mixture of Gaussian distributions).
- See also [Wu and Rosengren] for proper elements and ML clustering.
- Explosions of two Molniya-type satellites with $(a, e, i) = (15\,000, 0.1, 10^o)$, $(30\,000, e = 0.12, i = 22^o)$.

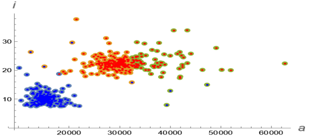
▷ `Kmeans` (unsupervised method, needing the number of clusters)



▷ `DBSCAN` (unsupervised method, not needing the number of clusters)



▷ `GaussianMixture` (gives 3 groups!).



Proper elements and clustering

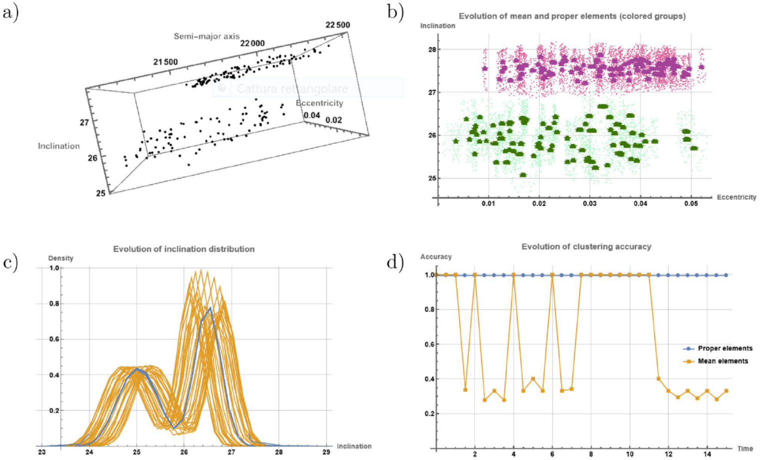


Fig. 3. (**a**) Initial distribution of all fragments in mean elements. (**b**) Evolution (in lighter colours) of the two groups (purple and green) in the mean (light) and proper (darker) colours for a period of 40 years. (**c**) The evolution of the estimated PDF in inclination over 40 years in mean (orange) and proper (blue) elements. (**d**) The evolution of the accuracy of DBSCAN for mean (orange) and proper (blue) elements.

Outline

Dissipative proper elements [A.C.,Gales,Lhotka,Vartolomei]

• For objects below 2000 km of altitude, one needs to consider the dissipative effect due to the atmospheric drag:

$$\dot a = -\frac{1}{2\pi}\int_0^{2\pi} B\,\rho\,v\frac{a}{1-e^2}\left[1+e^2+2e\cos f-\omega_E\cos i\sqrt{\frac{a^3(1-e^2)^3}{\mu_E}}\right]dM$$

$$\dot e = -\frac{1}{2\pi}\int_0^{2\pi} B\,\rho\,v\left[e+\cos f-\frac{r^2\omega_E\cos i}{2\sqrt{\mu_E a(1-e)^2}}\Big(2(e+\cos f)-e\sin^2 f\Big)\right]dM$$

▷ First (direct) method: build a canonical transformation in terms of Lie series, averaging the original Hamiltonian over the angle variables. This transformation is applied at each time step of the evolution of the mean elements. A single generating function per fragment is used for all integration times.

▷ Second (patching) method: partition the integration time into N subintervals and compute a generating function for each sub-interval via the direct transformation.

▷ Third (Brouwer-Hori) method: based on the transformation on the Hamiltonian part and also on the dissipative term.

▷ Fourth (Kamel) method: transform the dissipative system into a Hamiltonian one by doubling the phase space variables and compute the proper elements by eliminating its periodic terms.

Comparison between Brouwer-Hori, direct, patching

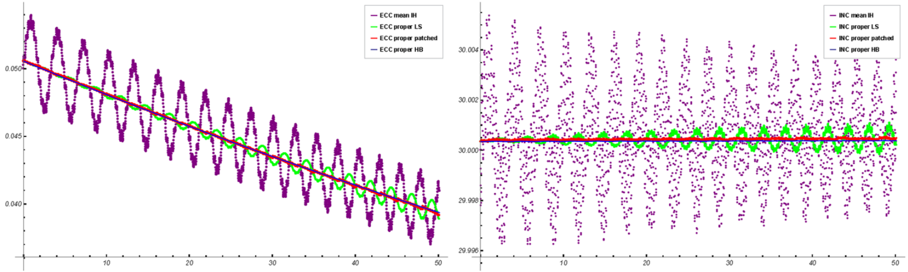


Figure: Comparison between the Brouwer-Hori (blue), direct (green) and patching (red) methods for $a = 7500$ km, $e = 0.05$, $i = 30^o$, $M = 0^o$, $\omega = 0^o$, $\Omega = 0^o$, $A/m = 1\text{m}^2/\text{kg}$ from [Celletti Gales Lhotka Vartolomei, 2025].

Recent references

- Celletti A., Pucacco G., Vartolomei T., "Reconnecting groups of space debris to their parent body through proper elements", Scientific Reports (2021)
- Celletti A., Pucacco G., Vartolomei T., "Proper elements for space debris", CM&DA (2022)
- Celletti A., Vartolomei T., "Old perturbative methods for a new problem in Celestial Mechanics: the space debris dynamics", BUMI (2023)
- Celletti A., Dogkas A., Vartolomei T., "Dynamics of highly eccentric and highly inclined space debris", CNSNS (2023)
- Apetrii M., Celletti A., Efthymiopoulos C., Gales C., Vartolomei T., "Simulating a breakup event and propagating the orbits of space debris", Celest. Mech. Dyn. Astron. (2024)
- Celletti A., Vartolomei T., "A dynamics based procedure for clustering and classifying space debris", Scientific Reports (2025)
- Celletti A., Vartolomei T., "Clustering space debris using perturbation theory", CNSNS (2025)
- Celletti A., Gales C., Lhotka C., Vartolomei T., "Analytical and computational methods for the determination of proper elements: an application to low Earth objects with dissipative drag", Preprint (2025)

CELMEC IX

• **CELMEC IX:** 14-18 September 2026, San Martino al Cimino, Viterbo, Italy, (flexible) deadline 31 March 2026
https://www.mat.uniroma2.it/~celmec/celmec9/

• **CELMEC prizes:** Topical issues in Cel. Mech. Dyn. Astron.
deadline 31 May 2026

▷ Analytical and semi-analytical results in Celestial Mechanics
▷ Pioneering computational techniques in Dynamical Astronomy

• Other on-going **Topical Collections** in Cel. Mech. Dyn. Astron.:

▷ Detection and Dynamics of Exoplanets (DDE): Interplay between theory and observations deadline 30 April 2026
▷ Machine Learning in Celestial Mechanics and Dynamical Astronomy, ongoing

Proper elements: procedure

✓ PROCEDURE:

1. Hamiltonian with Earth+Moon+Sun ⇒ average w.r.t. fast and semi-fast angles ⇒ the semimajor axis is constant and becomes the first proper element.
2. Fix reference values e_0 and i_0, and expand the Hamiltonian in power series around e_0, i_0 up to order 3.
3. Split the resulting Hamiltonian into the linear part and a remainder. Implement the normal form to remove the remainder to higher orders.
4. Once obtained the new normal form, disregard the remainder, so that the two actions corresponding to e and i become constants of motion.
5. The initial values of the new constants of motion, which are the two additional proper elements, are obtained back-transforming the canonical transformations in terms of the original variables, namely in terms of the initial data.

Proper elements: application

Celestial Mechanics and Dynamical Astronomy (2022) 134:11
https://doi.org/10.1007/s10569-022-10064-w

ORIGINAL ARTICLE

Proper elements for space debris

Alessandra Celletti[1] · Giuseppe Pucacco[2] · Tudor Vartolomei[1]

✓ APPLICATION:

1. simulate a break-up event with SIMPRO and obtain the Cartesian coordinates for all generated fragments;
2. compute the orbital elements for each fragment;
3. propagate each fragment for a given period of time to compute the osculating and/or mean elements
4. we implement the normal form to compute the proper elements of each fragment at the initial and final times;
5. proper and mean (or osculating) elements are compared at different times.

Brouwer-Hori method

Proposition

Consider the equations of motion:

$$\dot{\varphi}_i = \frac{\partial \mathcal{H}}{\partial J_i} + F_{\varphi,i} \ , \qquad \dot{J}_i = -\frac{\partial \mathcal{H}}{\partial \varphi_i} - F_{J,i} \ , \quad i = 1, 2 \ ,$$

for a Hamiltonian function $\mathcal{H}$ and non–gravitational contributions $(F_{\varphi,i}, F_{J,i})$. Under a canonical change of variables $(\underline{\varphi}, \underline{J}) \to (\underline{\varphi}', \underline{J}')$ the transformed equations are

$$\dot{\varphi}_i' = \frac{\partial \mathcal{H}'}{\partial J_i'} + F_{\varphi,i}' \ , \qquad \dot{J}_i' = -\frac{\partial \mathcal{H}'}{\partial \varphi_i} - F_{J,i}' \ , \quad i = 1, 2 \ ,$$

where

$$\begin{aligned} F_{\varphi,i}' &\equiv \underline{F}_\varphi \cdot \frac{\partial \underline{J}}{\partial J_i'} + \underline{F}_J \cdot \frac{\partial \underline{\varphi}}{\partial J_i'} \\ F_{J,i}' &\equiv \underline{F}_\varphi \cdot \frac{\partial \underline{J}}{\partial \varphi_i'} + \underline{F}_J \cdot \frac{\partial \underline{\varphi}}{\partial \varphi_i'} \ , \qquad i = 1, 2 \ . \end{aligned}$$

Kamel method

• Non-Hamiltonian vector field: $\underline{\dot{y}} = \underline{g}(\underline{y}, t)$, introduce an *adjoint* vector $\underline{Y} \in \mathbb{R}^n$ and a Hamiltonian K, defined as the projection of $\underline{g}$ on $\underline{Y}$:

$$K(\underline{y}, \underline{Y}, t) \equiv \underline{Y} \cdot \underline{g}(\underline{y}, t) \ . \tag{1}$$

Hamilton's equations associated to K are given by

$$\begin{aligned} \underline{\dot{y}} &= \frac{\partial K}{\partial \underline{Y}} = \underline{g}(\underline{y}, t) \\ \underline{\dot{Y}} &= -\frac{\partial K}{\partial \underline{y}} = -\underline{Y} \cdot \frac{\partial \underline{g}}{\partial \underline{y}} \ . \end{aligned}$$

• This *Hamiltonianization* procedure requires to double the order of the system, since the Hamiltonian K is $2n$-dimensional with an explicit time dependence. Once obtained the Hamiltonian K, we implement the standard normalization procedure.