“New techniques for studying the tidal evolution of planetary systems”

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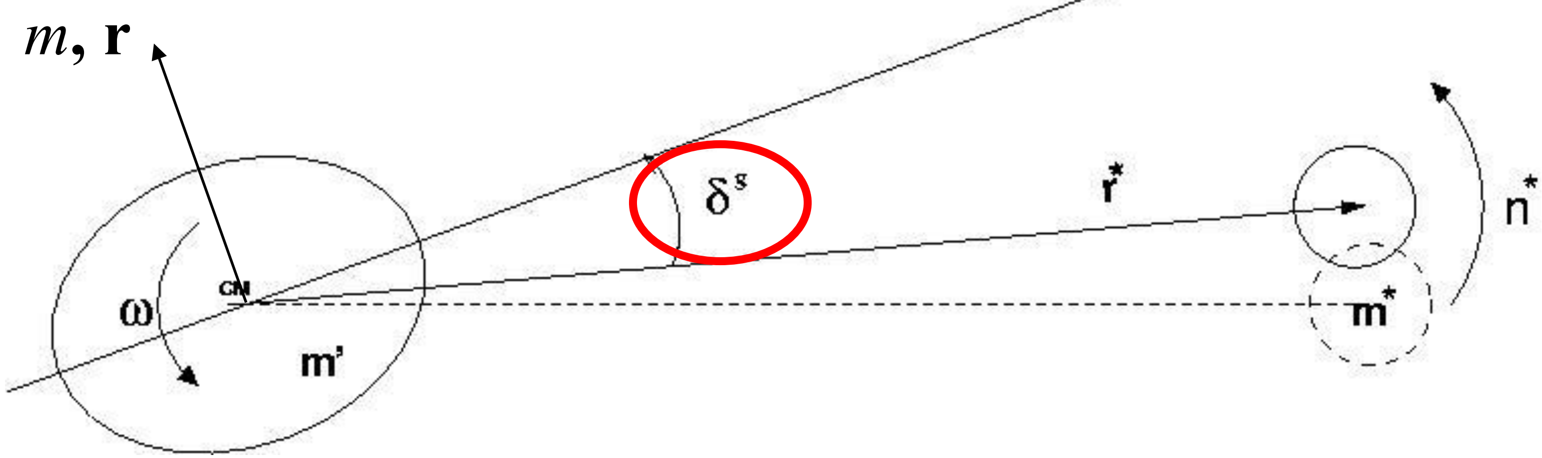
REPÚBLICA PORTUGUESA

Tides and motions in planetary system

Observatoire de Paris, 8-10 April 2026

Tidal potential

Darwin (1879), Love (1911), Kaula (1964)



$$U_g(\boldsymbol{r}) = -k_2 \frac{4\pi}{5} \frac{G m^* m R^5}{a^{*3} a^3} \sum_m \sum_{p,q,r,s} F^m_{pq}(e,i,\varepsilon) F^m_{rs}(e^*,i^*,\varepsilon) e^{\mathrm{i}(\nu^*_{mrs} - \sigma \Delta t^g(\sigma) - \nu_{mpq})}$$

$$\frac{d\omega}{dt} = -\frac{\partial}{\partial \theta}\left(\frac{\overline{U}}{C}\right)$$

$$\frac{da}{dt} = \frac{2}{\mu n a}\frac{\partial U}{\partial M}$$

equations of motion (orbit)

$$\dot{a}_{lmpq} = -\frac{2D_{lpq}}{a^{2l+2}}\left[\frac{a}{G(M+m)}\right]^{1/2}(l-2p+q)\Big[k_l a_e^{2l+1} B_{lm}(F_{lmp})^2 \sin \epsilon_{lmpq} + \frac{M}{m^*} k_l^* a_e^{*2l+1} B_{lm}^*(F_{lmp}^*)^2 \sin \epsilon_{lmpq}^*\Big]$$

$$\dot{e}_{lmpq} = -\frac{D_{lpq}}{ea^{2l+2}}\left[\frac{(1-e^2)}{aG(M+m)}\right]^{1/2}[(1-e^2)^{1/2}(l-2p+q)-(l-2p)] \cdot \Big[k_l a_e^{2l+1} B_{lm}(F_{lmp})^2 \sin \epsilon_{lmpq} + \frac{M}{m^*} k_l^* a_e^{*2l+1} B_{lm}^*(F_{lmp}^*)^2 \sin \epsilon_{lmpq}^*\Big]$$

$$= \frac{(1-e^2)^{1/2}}{2ae} \cdot \frac{[(1-e^2)^{1/2}(l-2p+q)-(l-2p)]}{l-2p+q}\dot{a}_{lmpq}, \quad (l-2p+q) \neq 0$$

$$\left[\frac{di}{dt}\right]_{lmpq} = -\frac{D_{lpq}}{a^{2l+2}}\left[\frac{1}{aG(M+m)(1-e^2)}\right]^{1/2} \cdot \Big[k_l a_e^{2l+1} B_{lm}(F_{lmp})^2 \frac{(l-2p)\cos i - m}{\sin i} \sin \epsilon_{lmpq} + \frac{M}{m^*} k_l^* a_e^{*2l+1} B_{lm}^*(F_{lmp}^*)^2 \frac{(l-2p)\cos i^* - m}{\sin i^*} \sin \epsilon_{lmpq}^*\Big] \quad (38)$$

possible singularities!

frame dependent!

Kaula (1964)

equations of motion (orbit)

$$\frac{\mathrm{d}i}{\mathrm{d}t} = -\frac{n}{\sqrt{1-e^2}}\frac{M'}{M}\sum_{l=2}^{\infty}\left(\frac{R}{a}\right)^{2l+1}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}(2-\delta_{0m})$$

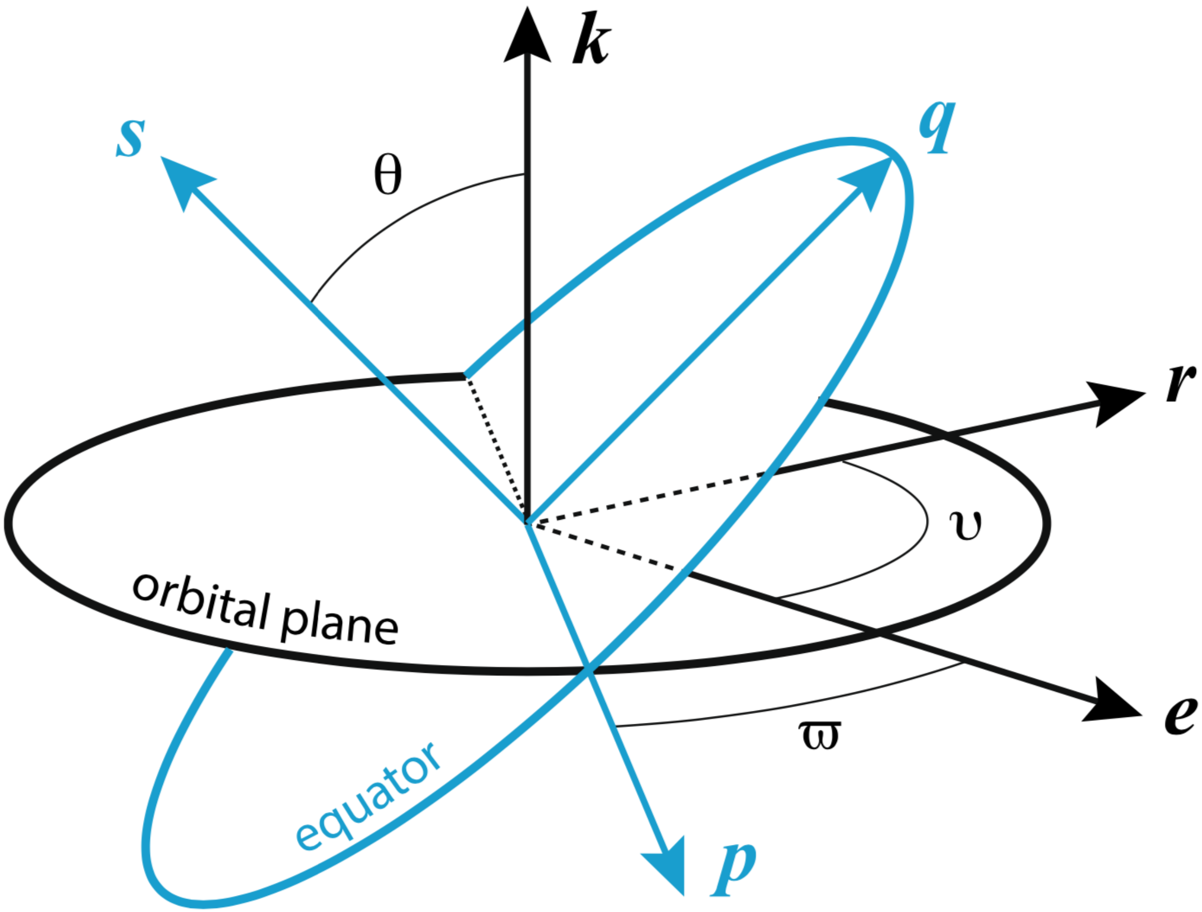
$$\sum_{p=0}^{l}\frac{(l-2p)\cos i - m}{\sin i}F^2_{lmp}(i)\sum_{q=-\infty}^{\infty}G^2_{lpq}(e)\,K_l(\omega_{lmpq})$$

$$+\frac{\beta n^2 a^2}{C\dot{\theta}}\frac{M'}{M}\sum_{l=2}^{\infty}\left(\frac{R}{a}\right)^{2l+1}\sum_{m=0}^{l}\frac{(l-m)!}{(l+m)!}(2-\delta_{0m})$$

$$\sum_{p=0}^{l}\frac{m\cos i-(l-2p)}{\sin i}F^2_{lmp}(i)\sum_{q=-\infty}^{\infty}G^2_{lpq}(e)\,K_l(\omega_{lmpq})\,.$$

Boué & Efroimsky (2019)

vectorial approach

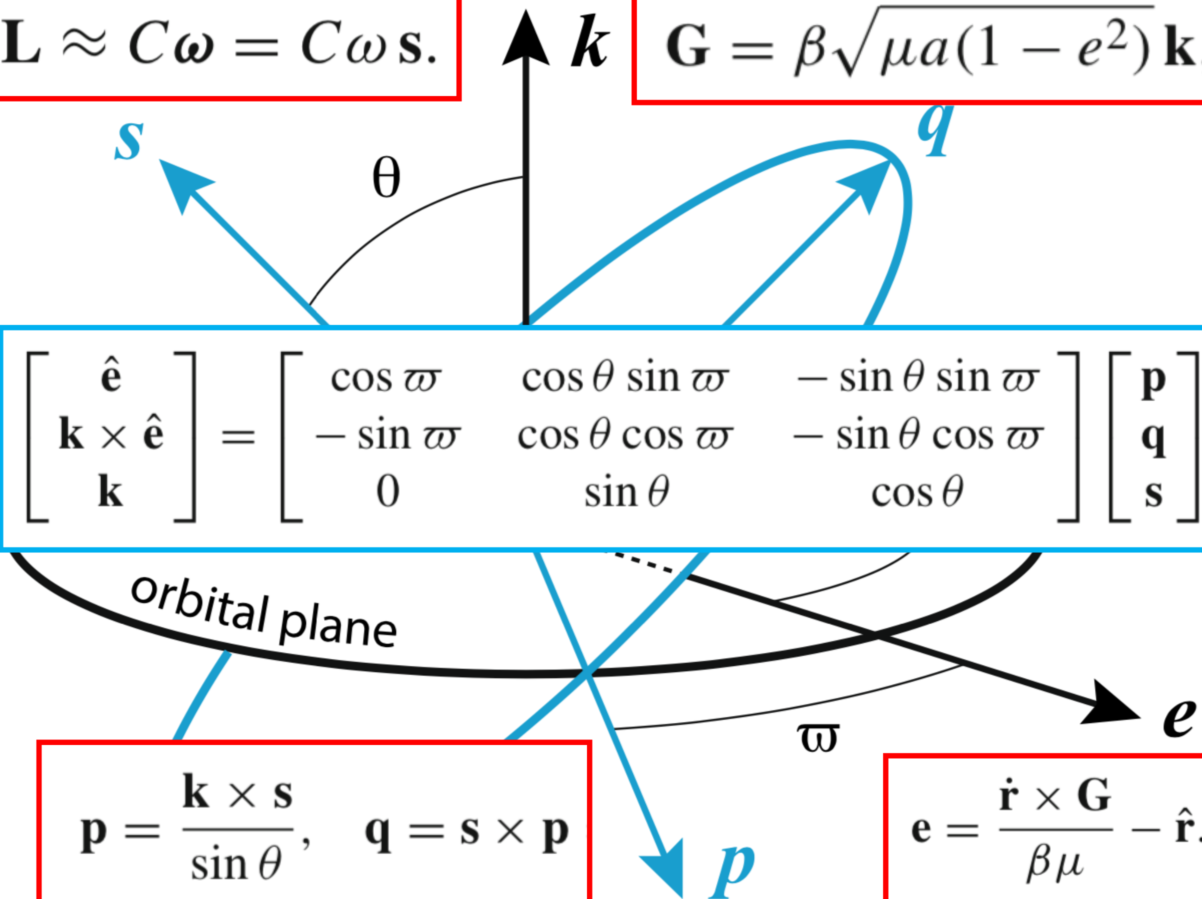


Correreia & Valente (2022)

vectorial approach

$$\mathbf{L} \approx C\boldsymbol{\omega} = C\omega\,\mathbf{s}.$$

$$\mathbf{G} = \beta\sqrt{\mu a(1-e^2)}\,\mathbf{k},$$



$$\begin{bmatrix} \hat{\mathbf{e}} \\ \mathbf{k}\times\hat{\mathbf{e}} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} \cos\varpi & \cos\theta\sin\varpi & -\sin\theta\sin\varpi \\ -\sin\varpi & \cos\theta\cos\varpi & -\sin\theta\cos\varpi \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{s} \end{bmatrix}$$

$$\mathbf{p} = \frac{\mathbf{k}\times\mathbf{s}}{\sin\theta}, \quad \mathbf{q} = \mathbf{s}\times\mathbf{p}$$

$$\mathbf{e} = \frac{\dot{\mathbf{r}}\times\mathbf{G}}{\beta\mu} - \hat{\mathbf{r}}.$$

Correia & Valente (2022)

vectorial approach
(quadrupole order)

$$V(\mathbf{r}) = -\frac{\mathcal{G}m}{r} + \Delta V(\mathbf{r}),$$

$$\mathcal{I} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix}$$

$$\Delta V(\mathbf{r}) = \frac{3\mathcal{G}}{2r^3}\,\hat{\mathbf{r}} \cdot \mathcal{I} \cdot \hat{\mathbf{r}},$$

e.g. Goldstein (1950)

incompressible body:

$$\mathrm{tr}(\mathcal{I}) = I_{11} + I_{22} + I_{33} = 0 \quad \Longrightarrow \quad \boxed{\mathbf{L} = C\boldsymbol{\omega} + \mathcal{I} \cdot \boldsymbol{\omega},}$$

small deformations:

$$I_{ij} \ll C \ (i, j = 1, 2, 3), \quad \Longrightarrow \quad \boxed{\mathbf{L} \approx C\boldsymbol{\omega} = C\omega\,\mathbf{s}.}$$

equations of motion (orbit)

potential energy: $$U(\mathbf{r}) = m_0 \Delta V(\mathbf{r})$$

$$U(\mathbf{r}) = \frac{3\mathcal{G}m_0}{2r^3}\Big[(I_{22} - I_{11})\big(\hat{x}_2^2 - \frac{1}{3}\big) + (I_{33} - I_{11})\big(\hat{x}_3^2 - \frac{1}{3}\big) + 2\big(I_{12}\hat{x}_1\hat{x}_2 + I_{13}\hat{x}_1\hat{x}_3 + I_{23}\hat{x}_2\hat{x}_3\big)\Big].$$

$$\hat{\mathbf{r}} = \mathbf{r}/r = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \qquad (\mathbf{p}, \mathbf{q}, \mathbf{s})$$

tidal force: $$\mathbf{F} = -\nabla U(\mathbf{r}) = \mathbf{F}_1 + \mathbf{F}_2$$

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^2}\hat{\mathbf{r}} + \frac{\mathbf{F}}{\beta}.$$

$$\mathbf{F}_1 = \frac{15\mathcal{G}m_0}{r^4}\Big[\frac{I_{22} - I_{11}}{2}\big(\hat{x}_2^2 - \frac{1}{5}\big) + \frac{I_{33} - I_{11}}{2}\big(\hat{x}_3^2 - \frac{1}{5}\big) + I_{12}\hat{x}_1\hat{x}_2 + I_{13}\hat{x}_1\hat{x}_3 + I_{23}\hat{x}_2\hat{x}_3\Big]\hat{\mathbf{r}},$$

$$\mathbf{F}_2 = -\frac{3\mathcal{G}m_0}{r^4}\Big[(I_{22} - I_{11})\hat{x}_2\,\mathbf{q} + (I_{33} - I_{11})\hat{x}_3\,\mathbf{s} + I_{12}(\hat{x}_1\,\mathbf{q} + \hat{x}_2\,\mathbf{p}) + I_{13}(\hat{x}_1\,\mathbf{s} + \hat{x}_3\,\mathbf{p}) + I_{23}(\hat{x}_2\,\mathbf{s} + \hat{x}_3\,\mathbf{q})\Big].$$

equations of motion

tidal torque:

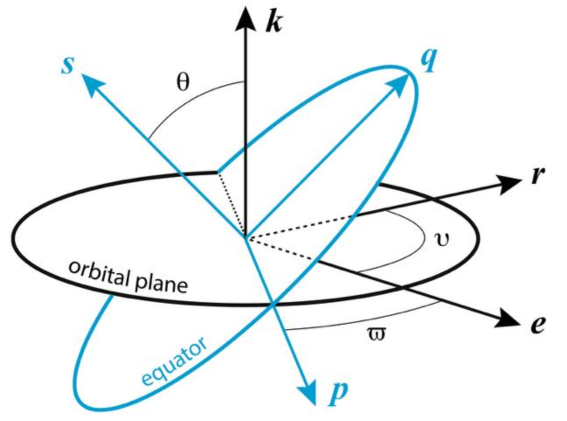
$$\dot{\mathbf{G}} = \mathbf{T} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F}_2,$$

$$\dot{\mathbf{L}} = -\dot{\mathbf{G}} = -\mathbf{T},$$

$$\begin{aligned}\mathbf{T} = -\frac{3\mathcal{G}m_0}{r^3}\Big\{ & \left[(I_{33} - I_{22})\hat{x}_2\hat{x}_3 - I_{12}\hat{x}_1\hat{x}_3 + I_{13}\hat{x}_1\hat{x}_2 + I_{23}(\hat{x}_2^2 - \hat{x}_3^2)\right]\mathbf{p} \\ & + \left[(I_{11} - I_{33})\hat{x}_1\hat{x}_3 + I_{12}\hat{x}_2\hat{x}_3 + I_{13}(\hat{x}_3^2 - \hat{x}_1^2) - I_{23}\hat{x}_1\hat{x}_2\right]\mathbf{q} \\ & + \left[(I_{22} - I_{11})\hat{x}_1\hat{x}_2 + I_{12}(\hat{x}_1^2 - \hat{x}_2^2) - I_{13}\hat{x}_2\hat{x}_3 + I_{23}\hat{x}_1\hat{x}_3\right]\mathbf{s}\Big\}.\end{aligned}$$

secular evolution

$$\dot{\mathbf{e}} = \frac{1}{\beta\mu}\left(\frac{\mathbf{F}}{\beta} \times \mathbf{G} + \dot{\mathbf{r}} \times \mathbf{T}\right).$$



equations of motion

tidal torque:

$$\dot{\mathbf{G}} = \mathbf{T} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F}_2,$$

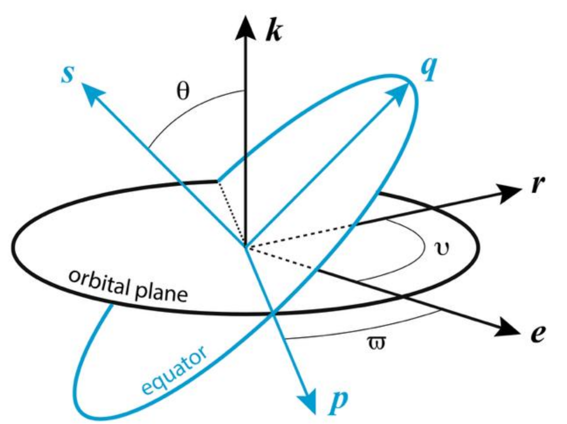
$$\dot{\mathbf{L}} = -\dot{\mathbf{G}} = -\mathbf{T},$$

$$\mathbf{T} = -\frac{3\mathcal{G}m_0}{r^3}\Big\{\left[(I_{33}-I_{22})\hat{x}_2\hat{x}_3 - I_{12}\hat{x}_1\hat{x}_3 + I_{13}\hat{x}_1\hat{x}_2 + I_{23}(\hat{x}_2^2 - \hat{x}_3^2)\right]\mathbf{p}$$
$$+\left[(I_{11}-I_{33})\hat{x}_1\hat{x}_3 + I_{12}\hat{x}_2\hat{x}_3 + I_{13}(\hat{x}_3^2 - \hat{x}_1^2) - I_{23}\hat{x}_1\hat{x}_2\right]\mathbf{q}$$
$$+\left[(I_{22}-I_{11})\hat{x}_1\hat{x}_2 + I_{12}(\hat{x}_1^2 - \hat{x}_2^2) - I_{13}\hat{x}_2\hat{x}_3 + I_{23}\hat{x}_1\hat{x}_3\right]\mathbf{s}\Big\}.$$

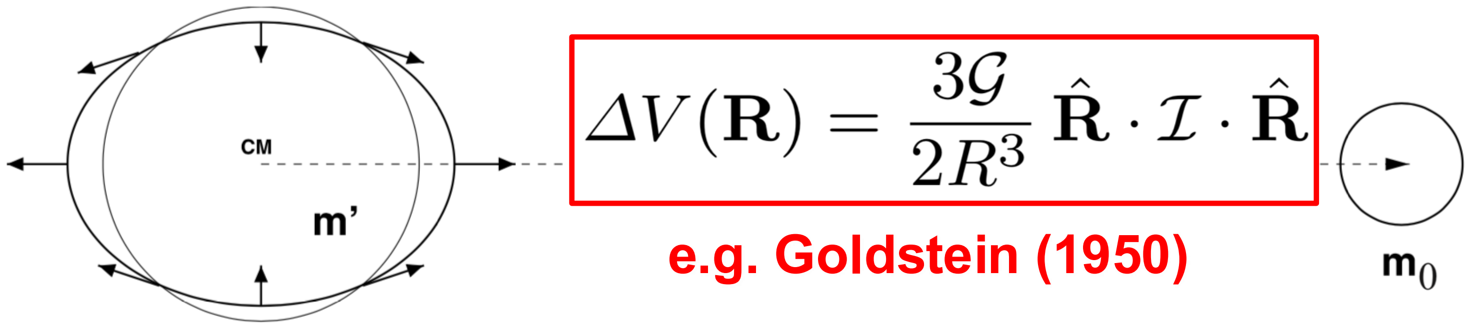
missing information

secular evolution

$$\dot{\mathbf{e}} = \frac{1}{\beta\mu}\left(\frac{\mathbf{F}}{\beta} \times \mathbf{G} + \dot{\mathbf{r}} \times \mathbf{T}\right).$$



tidal deformation
(time domain)



$$\Delta V(\mathbf{R}) = \frac{3\mathcal{G}}{2R^3}\,\hat{\mathbf{R}}\cdot\mathcal{I}\cdot\hat{\mathbf{R}}$$

e.g. Goldstein (1950)

tidal potential (Love 1911):

Love distribution (k_2)

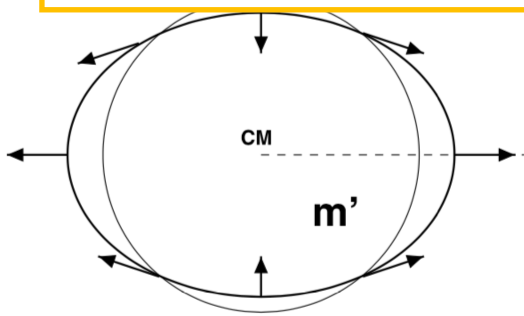
$$\Delta V(\mathbf{R},t) = \int_{-\infty}^{0} k_2(t-t')V_p(\mathbf{R},t')\,dt' = k_2(t) * V_p(\mathbf{R},t)$$

perturbing potential (e.g. Kaula 1964):

$$V_p(\mathbf{R}) = -\frac{\mathcal{G}m_0}{r}\left(\frac{R}{r}\right)^2 P_2(\hat{\mathbf{R}}\cdot\hat{\mathbf{r}}),$$

tidal deformation

$$\mathcal{I}^B(t) = \int k_2(t-t')\,\mathcal{I}_p^B(t')\,\mathrm{d}t' = k_2(t) * \mathcal{I}_p^B(t),$$



$$\Delta V(\mathbf{R}) = \frac{3\mathcal{G}}{2R^3}\,\hat{\mathbf{R}}\cdot\mathcal{I}\cdot\hat{\mathbf{R}}$$

e.g. Goldstein (1950)

$\mathbf{m}_0$

tidal potential (Love 1911):

Love distribution (k_2)

$$\Delta V(\mathbf{R},t) = \int_{-\infty}^{0} k_2(t-t')\,V_p(\mathbf{R},t')\,dt' = k_2(t) * V_p(\mathbf{R},t)$$

perturbing potential (Correia & Valente 2022):

$$V_p(\mathbf{R}) = \frac{3\mathcal{G}}{2R^3}\,\hat{\mathbf{R}}\cdot\mathcal{I}_p\cdot\hat{\mathbf{R}},$$

$$\frac{\mathcal{I}_p}{mR^2} = -\frac{m_0}{m}\left(\frac{R}{r}\right)^3\left(\hat{\mathbf{r}}\,\hat{\mathbf{r}}^T - \frac{1}{3}\mathbb{I}\right),$$

tidal deformation
(frequency domain)

$$\hat{\mathcal{I}}(\omega) = \int \mathcal{I}(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, dt = \hat{k}_2(\omega)\, \hat{\mathcal{I}}_p(\omega)$$

Love number

$$\hat{\mathcal{I}}_p(\omega) = \int \mathcal{I}_p(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, dt$$

$$\hat{k}_2(\omega) = \int k_2(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, dt$$

Deformation = | k_2 |

Dissipation = $b(\omega)$

$$\hat{k}_2(\omega) = a(\omega) - \mathrm{i}\, b(\omega)$$

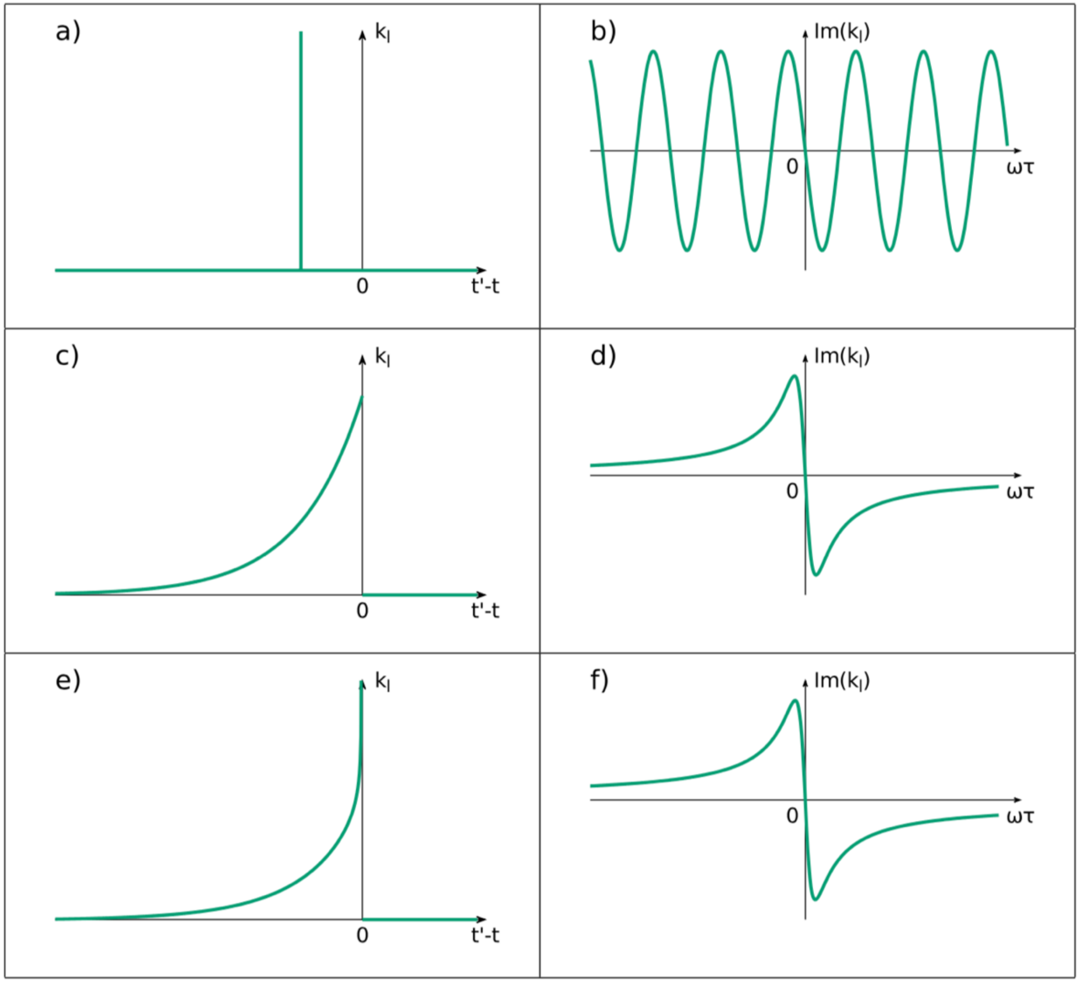
k_2 depends on the internal structure of the body and so we need to adopt a tidal model (rheology)

tidal models

constant time-lag:
(linear model)

Maxwell model:

Andrade model:

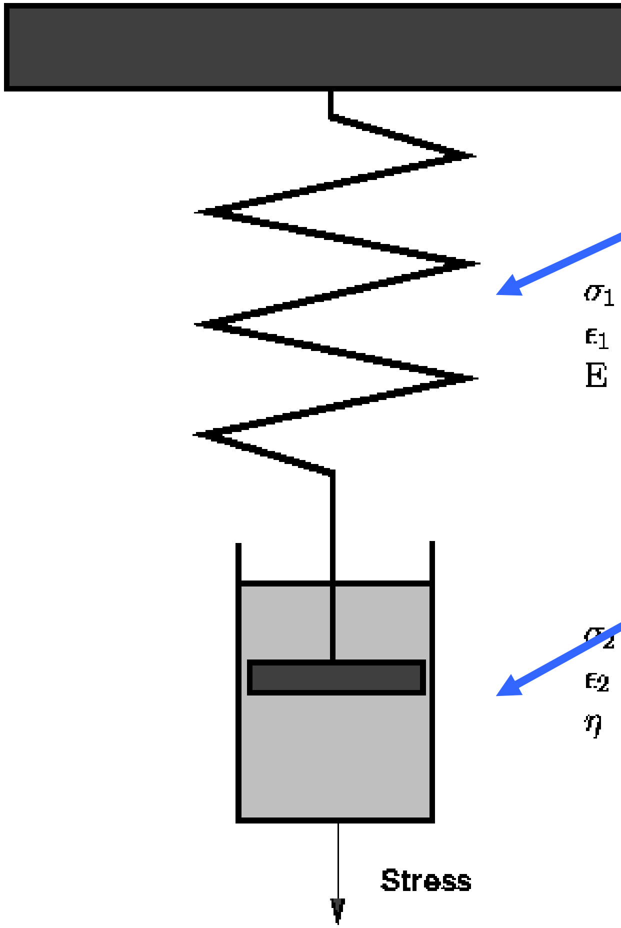


time

frequency

Boué et al. (2019)

Maxwell model (1867)



elastic

stress $\sigma = E\,\epsilon$ strain

viscous

$$\sigma = \eta\, d\epsilon/dt$$

$$\frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} = \frac{d\epsilon}{dt}$$

Frequency Domain

$$\hat{\mathcal{I}}(\omega) = \hat{k}_2(\omega)\,\hat{\mathcal{I}}_p(\omega)$$

$$\hat{k}_2(\omega) = \frac{k_{\mathrm{f}}}{1 + \hat{\mu}(\omega)} \qquad \hat{\mu}(\omega) \equiv \hat{\sigma}(\omega)/\hat{\varepsilon}(\omega)$$

$$\sigma(\omega) = \sigma_0\,\mathrm{e}^{i\omega t} \quad ; \quad \epsilon(\omega) = \epsilon_0\,\mathrm{e}^{i\omega t}$$

$$\frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\eta} = \frac{d\epsilon}{dt}$$

$$\tau_e = \eta/E \quad ; \quad \tau = \eta(1+E)/E$$

Frequency Domain

$$\hat{\mathcal{I}}(\omega) = \hat{k}_2(\omega)\,\hat{\mathcal{I}}_p(\omega)$$

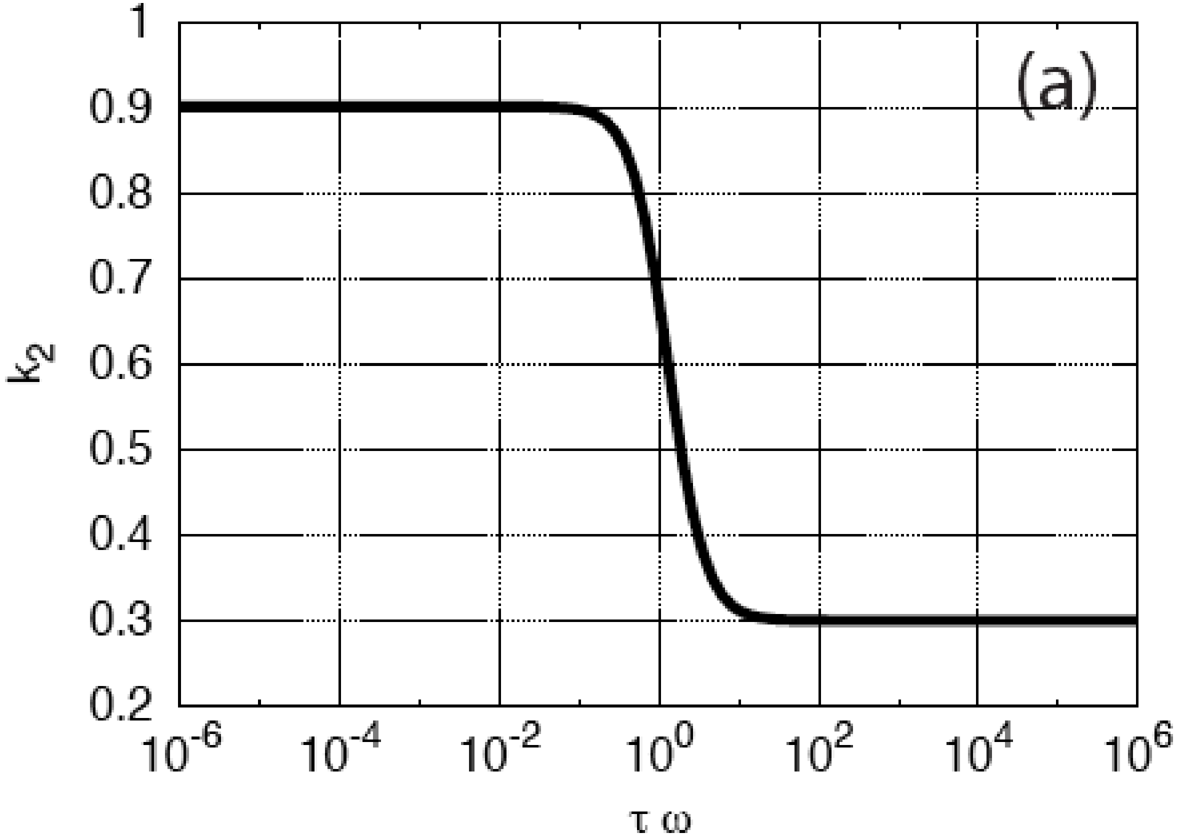
$$\hat{k}_2(\omega) = \frac{k_\mathrm{f}}{1+\hat{\mu}(\omega)} \qquad \hat{\mu}(\omega) \equiv \hat{\sigma}(\omega)/\hat{\varepsilon}(\omega)$$

$$\hat{k}_2(\omega) = k_\mathrm{f}\,\frac{1+\mathrm{i}\omega\tau_e}{1+\mathrm{i}\omega\tau}$$

$$\tau_e = \eta/E \quad ; \quad \tau = \eta(1+E)/E$$

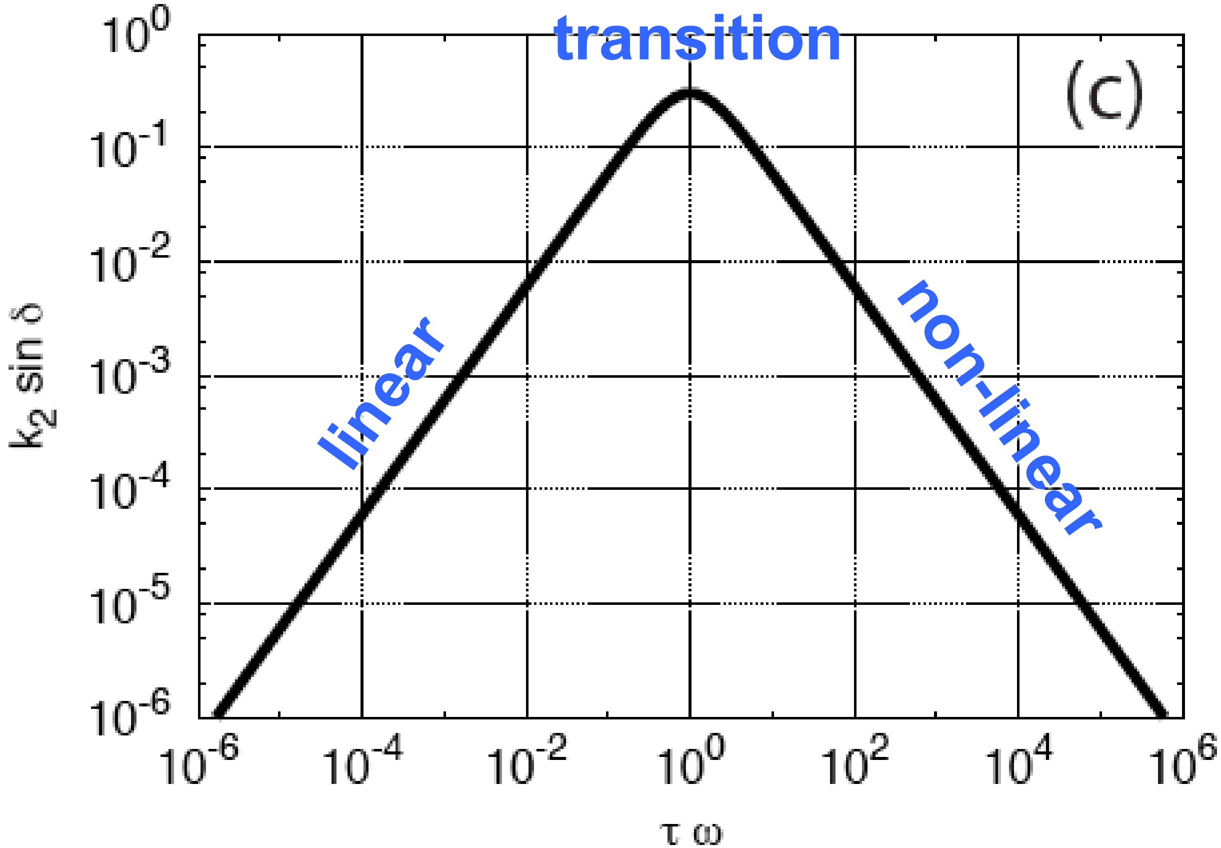
Frequency Domain

$$\hat{\mathcal{I}}(\omega) = k_{\mathrm{f}} \frac{1 + \mathrm{i}\omega\tau_e}{1 + \mathrm{i}\omega\tau} \hat{\mathcal{I}}_p(\omega)$$



Frequency Domain

$$\hat{\mathcal{I}}(\omega) = k_\mathrm{f}\,\frac{1+\mathrm{i}\omega\tau_e}{1+\mathrm{i}\omega\tau}\,\hat{\mathcal{I}}_p(\omega)$$



Frequency Domain

$$\hat{\mathcal{I}}(\omega) = k_{\mathrm{f}} \frac{1 + \mathrm{i}\omega\tau_e}{1 + \mathrm{i}\omega\tau} \hat{\mathcal{I}}_p(\omega)$$

$$(1 + \mathrm{i}\omega\tau)\,\hat{\mathcal{I}}(\omega) = (1 + \mathrm{i}\omega\tau_e)\,k_{\mathrm{f}}\hat{\mathcal{I}}_p(\omega)$$

Time Domain

$$\mathrm{i}\omega \to \frac{d}{dt}$$

$$\mathcal{I}(t) + \tau\,\dot{\mathcal{I}}(t) = k_{\mathrm{f}}\left(\mathcal{I}_p(t) + \tau_e\,\dot{\mathcal{I}}_p(t)\right)$$

time domain

- Complete model (captures the entire dynamics);
- Able to handle chaotic and transient events;
- Only feasible for simple tidal models (e.g., linear, Maxwell);
- Slow evolution (though same time-scale of n-body codes).

frequency domain

- Works for any rheological model;
- Fast evolution (suitable for Gyr time-scales);
- Secular model (misses fast perturbations);
- We need to truncate series in eccentricity;
- Only applies to periodic perturbations.

direct equations of motion
(time-domain)

orbit

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^2}\,\hat{\mathbf{r}} + \frac{\mathbf{F}}{\beta}.$$

$$\dot{\mathbf{L}} = -\dot{\mathbf{G}} = -\mathbf{T},$$

spin

$$\mathbf{F}_1 = \frac{15\mathcal{G}m_0}{r^4}\left[\frac{I_{22}-I_{11}}{2}\left(\hat{x}_2^2-\frac{1}{5}\right)+\frac{I_{33}-I_{11}}{2}\left(\hat{x}_3^2-\frac{1}{5}\right)+I_{12}\hat{x}_1\hat{x}_2+I_{13}\hat{x}_1\hat{x}_3+I_{23}\hat{x}_2\hat{x}_3\right]\hat{\mathbf{r}},$$

$$\mathbf{F}_2 = -\frac{3\mathcal{G}m_0}{r^4}\Big[(I_{22}-I_{11})\hat{x}_2\,\mathbf{q}+(I_{33}-I_{11})\hat{x}_3\,\mathbf{s} + I_{12}(\hat{x}_1\,\mathbf{q}+\hat{x}_2\,\mathbf{p})+I_{13}(\hat{x}_1\,\mathbf{s}+\hat{x}_3\,\mathbf{p})+I_{23}(\hat{x}_2\,\mathbf{s}+\hat{x}_3\,\mathbf{q})\Big].$$

$$\mathbf{T} = -\frac{3\mathcal{G}m_0}{r^3}\Big\{\left[(I_{33}-I_{22})\hat{x}_2\hat{x}_3-I_{12}\hat{x}_1\hat{x}_3+I_{13}\hat{x}_1\hat{x}_2+I_{23}(\hat{x}_2^2-\hat{x}_3^2)\right]\mathbf{p} + \left[(I_{11}-I_{33})\hat{x}_1\hat{x}_3+I_{12}\hat{x}_2\hat{x}_3+I_{13}(\hat{x}_3^2-\hat{x}_1^2)-I_{23}\hat{x}_1\hat{x}_2\right]\mathbf{q} + \left[(I_{22}-I_{11})\hat{x}_1\hat{x}_2+I_{12}(\hat{x}_1^2-\hat{x}_2^2)-I_{13}\hat{x}_2\hat{x}_3+I_{23}\hat{x}_1\hat{x}_3\right]\mathbf{s}\Big\}.$$

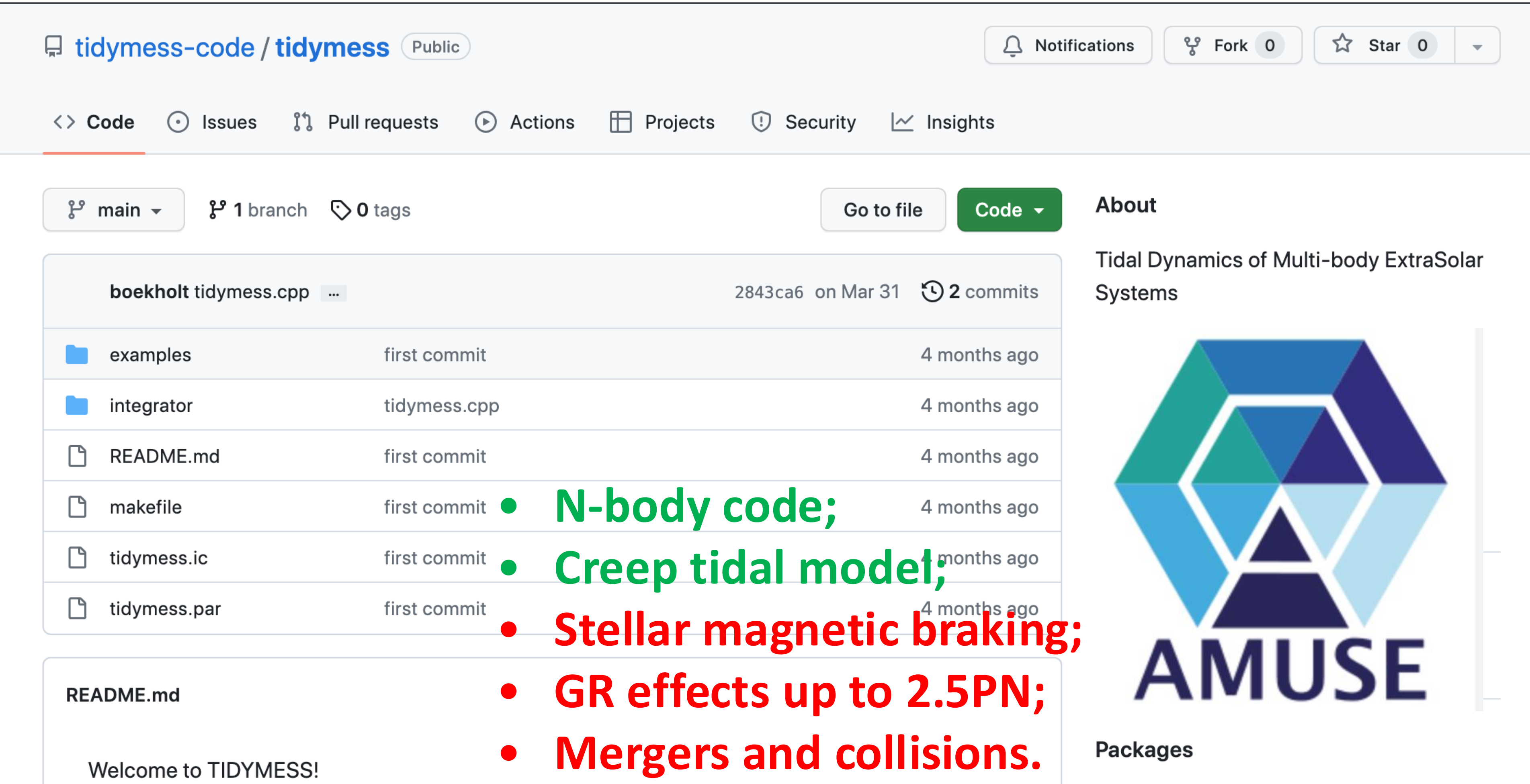
Tidymess N-body code
(Boekholt & Correia 2023)

$$\mathcal{I}(t)+\tau\,\dot{\mathcal{I}}(t) = k_{\mathrm{f}}\left(\mathcal{I}_p(t)+\tau_e\,\dot{\mathcal{I}}_p(t)\right)$$

deformation

tidymess

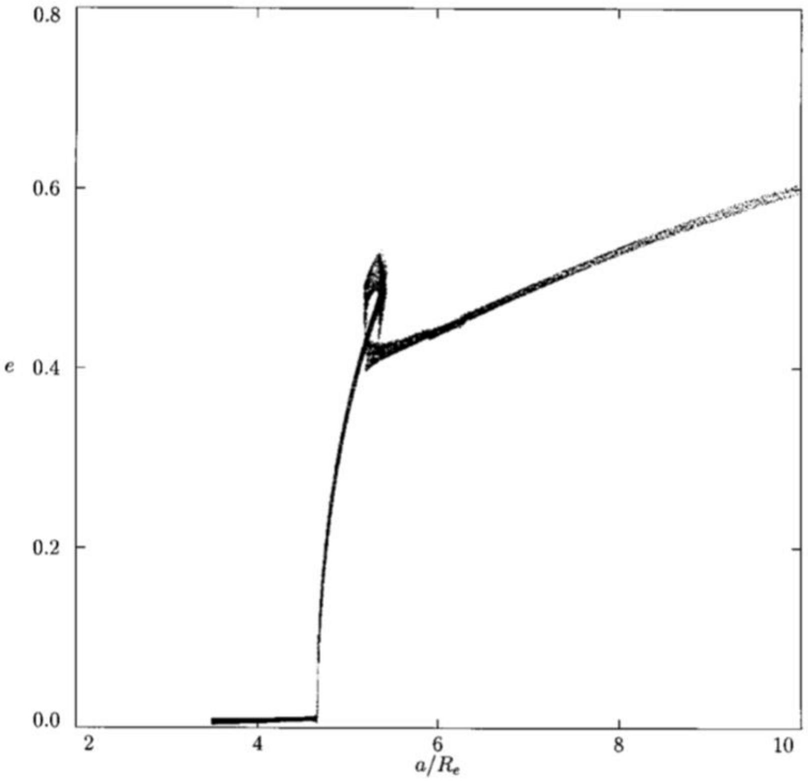
https://github.com/tidymess-code

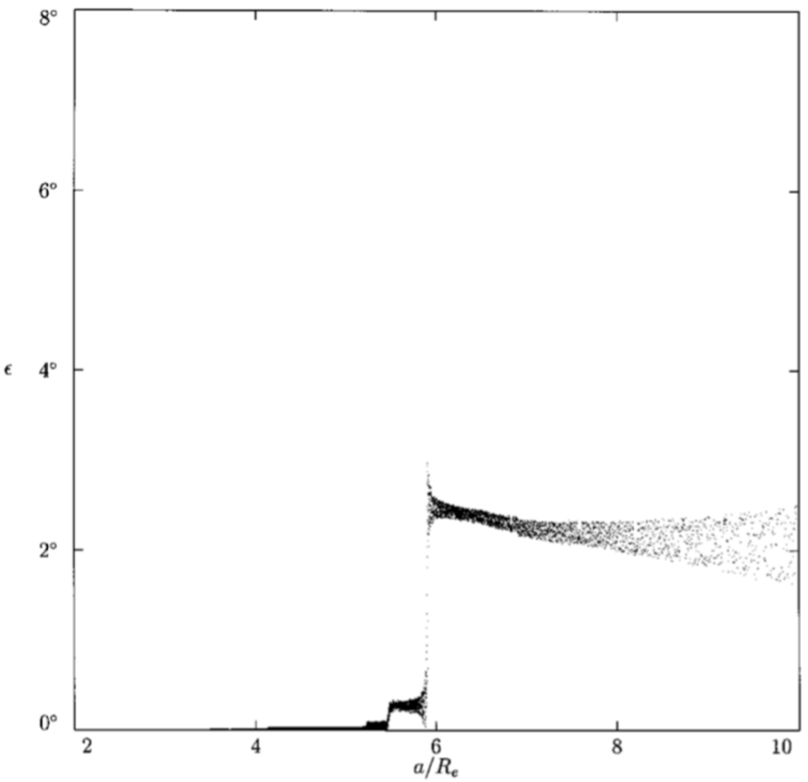


- N-body code;
- Creep tidal model;
- Stellar magnetic braking;
- GR effects up to 2.5PN;
- Mergers and collisions.

constant (symplectic), adaptive, weighted, and symmetrized time-steps

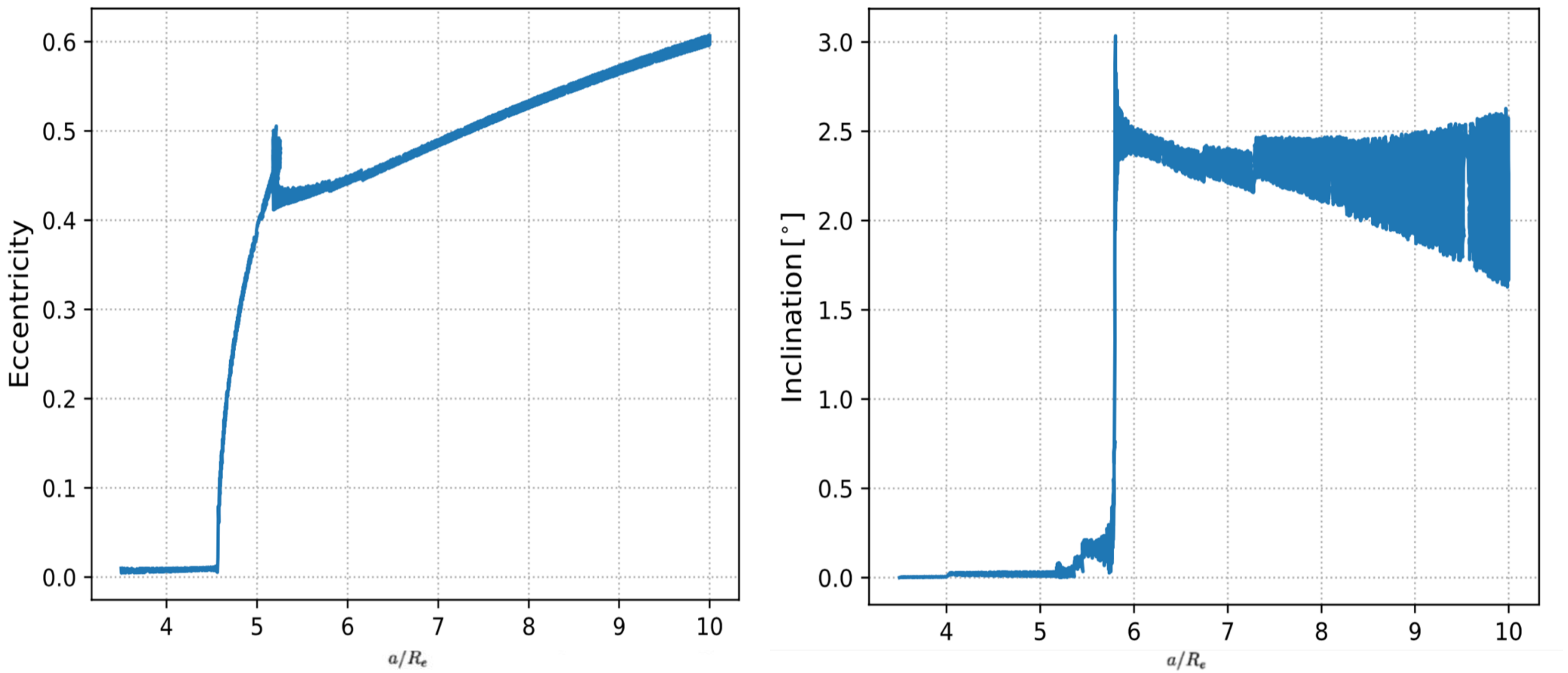
Earth-Moon early evolution
(time-domain)





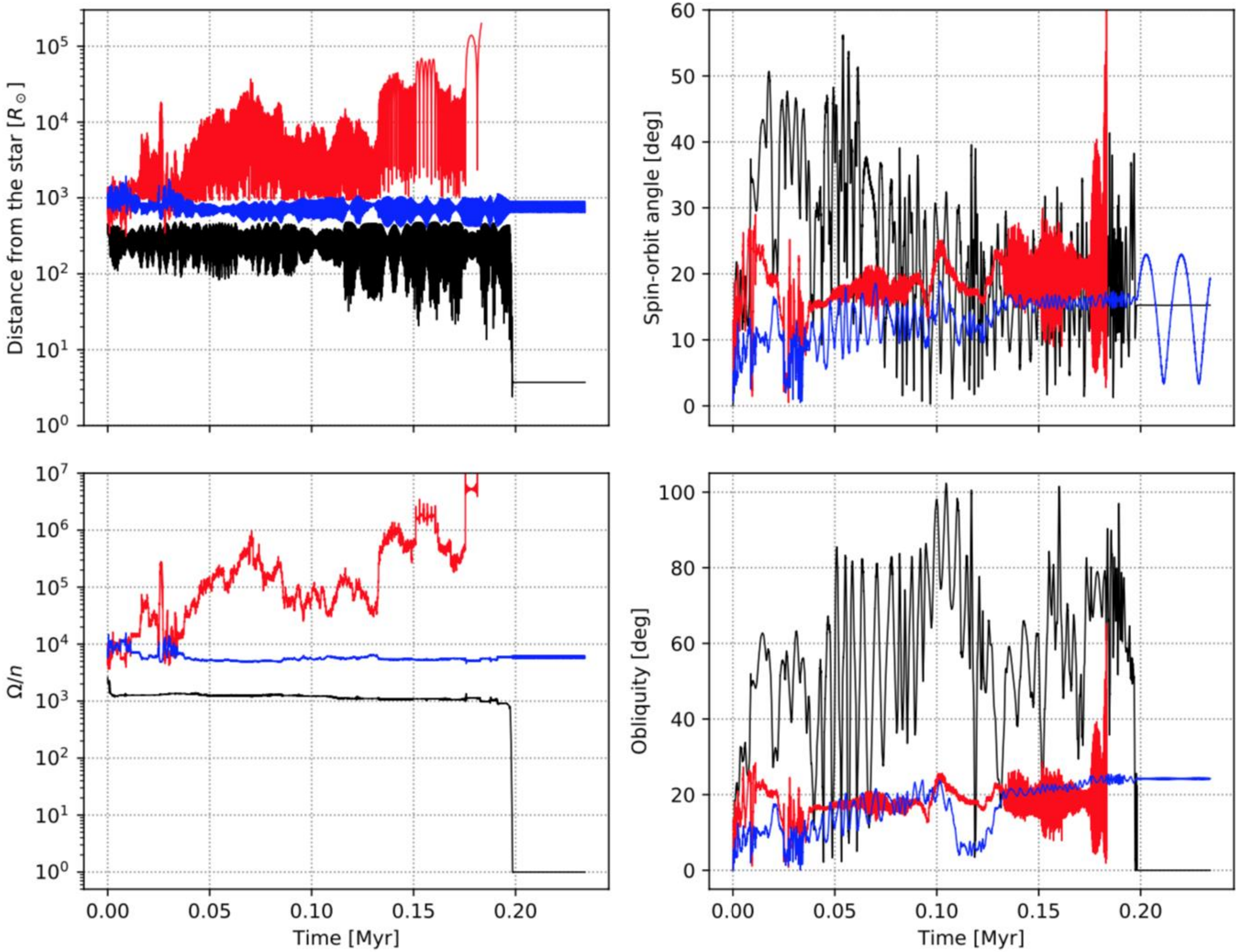
Touma & Wisdom (1998)

Earth-Moon early evolution
(time-domain)

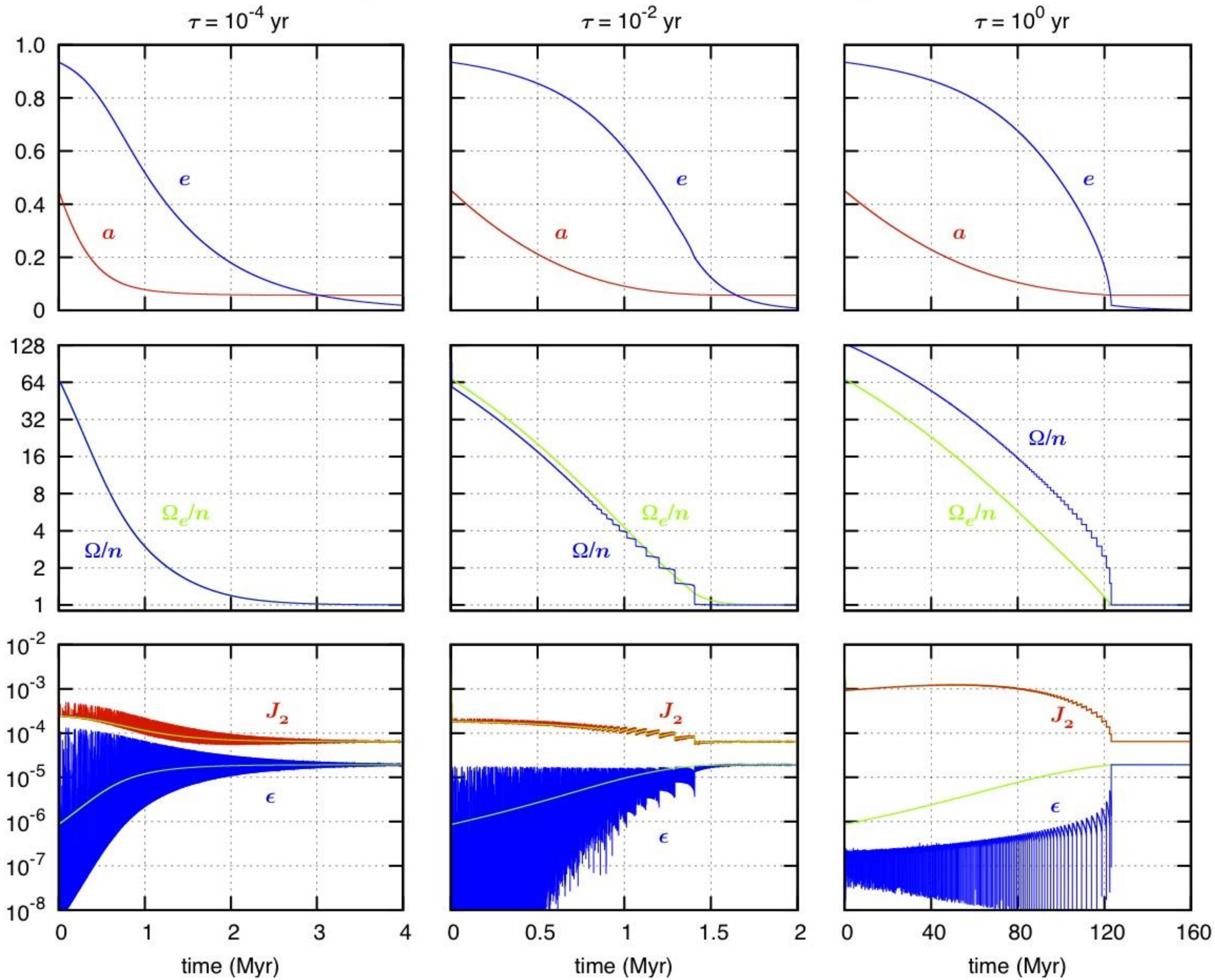


Tidymess N-body code
(creep non-linear model)

planet-planet scattering
(time-domain)



hot-Jupiter formation
(time-domain)



Conclusions

- **Tidal deformation and consequent dissipation result in spin and orbital evolution of planetary and stellar systems. We have revisited the tidal problem with different approaches.**
- **We compute the instantaneous deformation of extended bodies using a differential equation for the inertia tensor. This method can take into account a wide class of perturbations, including chaotic motion and transient events.**
- **We derive the equations of motion in a vectorial formalism, which is frame independent and valid for any rheological model. The vector basis depends only on the unit vectors of the spin and orbital angular momenta, which are easy to obtain and independent of the chosen frame.**