New synergies between traditional PN and EFT: contributions of the logarithmic tails in the energy

François Larrouturou
in collaboration with : L. Blanchet, S. Foffa & R. Sturani

Institut d'Astrophysique de Paris

based on *arXiv:1912.12359*

2ème rencontre du groupe de travail *Formes d'ondes* du GrD OG
Observatoire de Paris–Meudon, 4. February 2020

Groupement de recherche
Ondes gravitationnelles

SORBONNE UNIVERSITÉ
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Motivations

- The "tail" effects are very peculiar features of General Relativity[1]

↪ due to non-linearities, they have dissipative and conservative sectors,

↪ they take into account the whole history of the source,

↪ complicated to deal with: either use a peculiar action[2] or involving iterations of Einstein's equations[3].

Can we find a generic way to deal with tails ?

Spoiler: YES[a] !

↪ at least for the logarithmic effects of simple tails,

↪ using synergies between traditional PN methods and EFT,

↪ conserved energy computed for circular orbits up to 7PN.

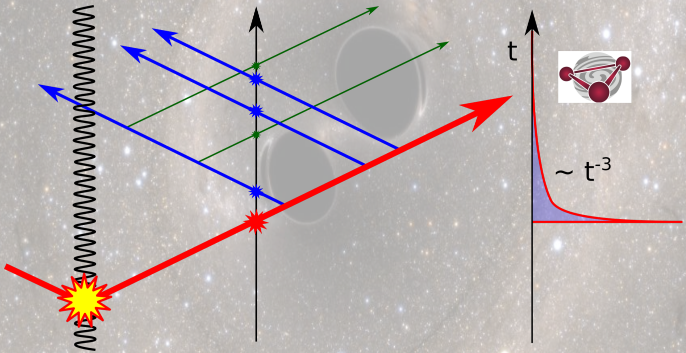
[a] L. Blanchet, S. Foffa, FL & R. Sturani, *arXiv:1912.12359*.

[1] see *eg.* L. Blanchet & T.Damour, PRD 37(1988)1410.
[2] see *eg.* S. Foffa & R. Sturani, PRD 87(2012)044056.
[3] see *eg.* A. Le Tiec, L. Blanchet & B. Whiting, PRD 85(2012)064039.

Contents

The radiative tails

- Tails are the effects of the back-scattering of the GWs on the (quasi-)static curvature.



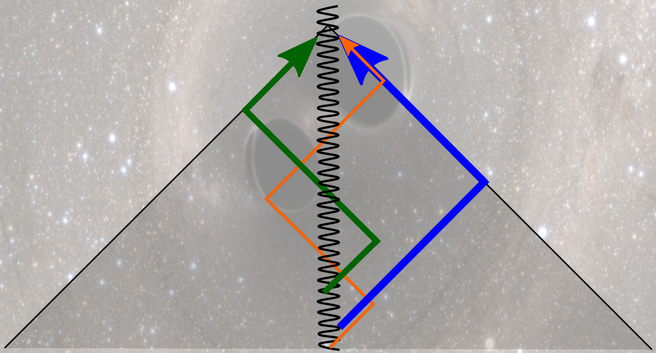
↪ Their effects is known up to tails-of-tails-of-tails[4] (4.5PN).

[4] T. Marchand, L. Blanchet & G. Faye, CQG 33(2016)244.

Effects of the tails in the energy

- From the point of view of the source, GWs scatter and "kick" back

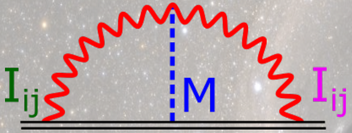
⇒ modification of the energy (conservative effect).



↪ Shows up at 4PN in the energy.

Why the log ? Hand-wawing argument

- Let's consider a simple tail



$$\Rightarrow \quad S \sim \int \mathrm{d}t \left\{ M I_{ij}(t) \int \frac{\mathrm{d}t'}{|t-t'|} I_{ij}^{(6)}(t') \right\}.$$

- The log roughly comes from the propagation of the GWs

$$\int \frac{\mathrm{d}t'}{|t-t'|} I_{ij}^{(6)}(t') \quad \rightsquigarrow \quad \int \mathrm{d}t' \, \log|t'-t| \, I_{ij}^{(7)}(t').$$

A general action for the logs in simple tails

The simple tail's log contributions are described by the non-local action

In EFT language (*ie.* Fourier domain)

$$S = -2\sum_{\ell=2}^{\infty}\frac{G^2 M}{c^{2\ell+4}}\int\frac{\mathrm{d}k_0}{2\pi}\log\left(\frac{|k_0|}{\mu}\right)k_0^{2\ell+2}\left[a_\ell\left|\hat{I}_L(k_0)\right|^2+\frac{b_\ell}{c^2}\left|\hat{J}_L(k_0)\right|^2\right]$$

In traditional PN language (*ie.* temporal domain)

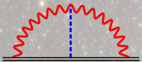
$$S = \sum_{\ell=2}^{\infty}\frac{G^2 M}{c^{2\ell+4}}\underset{\tau_0}{\mathrm{Pf}}\iint\frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|}\left[a_\ell I_L^{(\ell+1)}(t)I_L^{(\ell+1)}(t')+\frac{b_\ell}{c^2}J_L^{(\ell+1)}(t)J_L^{(\ell+1)}(t')\right]$$

The two scales are related by $c\tau_0 = e^{-\gamma_E}/\mu$, and the a_ℓ and b_ℓ read

$$a_\ell = \frac{(\ell+1)(\ell+2)}{\ell(\ell-1)\ell!(2\ell+1)!!} \quad \text{and} \quad b_\ell = \left(\frac{2\ell}{\ell+1}\right)^2 a_\ell.$$

Proof of the action: EFT's way

- As shown in [5], there is an intrinsic link between the singular, logarithmic and imaginary sectors of the simple tails and the GW emission-absorption process :



$$\ni -\frac{GM\,k_0}{c^2}\left[\frac{1}{d-3} + 2\log\left(\frac{|k_0|}{\mu}\right) - i\,\pi\right]\times \mathrm{Im}$$

With

$$\mathcal{A}_{ea}(k_0) = -\sum_{\ell=2}^{\infty}\frac{4\pi G}{c^{2\ell+2}\ell!}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k_{L-2}\,k_{L'-2}}{k^2-k_0^2-i\varepsilon}\Bigg(\Pi_I^{ijkl}I_{ij(L-2)}(k_0)I_{kl(L'-2)}(-k_0) + \frac{16\Pi_J^{ijkl}}{9c^2}J_{ij(L-2)}(k_0)J_{kl(L'-2)}(-k_0)\Bigg)\,,$$

where Π_I^{ijkl} and Π_J^{ijkl} encode appropriate polarizations, and depend on k_μ.

$\Rightarrow$ Computing it, one recovers the action presented.

[5] S. Foffa & R. Sturani, *arXiv:1907.02869*.

Proof of the action: traditional PN's way

- S has also been derived by traditional PN methods, up to 1PN.

$\hookrightarrow$ How to do that ? The hard way !

$\Rightarrow$ Iterate the vacuum Einstein's equations to have the tail part of the metric, and expand it in the near-zone (*ie.* $r \to 0$) limit, *eg.*

$$h^{00}_{2,\text{tail}} = \frac{8G^2M}{c^5} \log\left(\frac{\omega r}{2\pi c}\right) \sum_{\ell=2}^{\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[\frac{I_L^{(1)}(t - r/c) - I_L^{(1)}(t + r/c)}{r} \right],$$

$\Rightarrow$ apply a convenient gauge transformation, so $\tilde{h}^{ij}_{2,\text{tail}} \ll \tilde{h}^{0i}_{2,\text{tail}} \ll \tilde{h}^{00}_{2,\text{tail}}$,

$\Rightarrow$ compute the 1PN relative acceleration of a test particle *via*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{g_{i\mu}\, v^\mu}{\sqrt{-g_{\mu\nu}\frac{v^\mu v^\nu}{c^2}}}\right] = \frac{1}{2}\frac{v^\mu v^\nu}{\sqrt{-g_{\mu\nu}\frac{v^\mu v^\nu}{c^2}}}\,\partial_i g_{\mu\nu}\,,$$

$\Rightarrow$ derive the action which variation reads $\delta S = \int \mathrm{d}t\, m\, a^i\, \delta y^i$.

Contribution of the simple tails: general orbits

- Defining $\mathcal{I}_L(t) = \underset{\tau_0}{\mathrm{Pf}} \int \frac{\mathrm{d}t'}{|t-t'|}\, I_L(t')$, the logarithmic tail contribute as

$$\Delta E_{\text{tails}} = \sum_{\ell=2}^{\infty} \frac{G^2 a_\ell}{c^{2\ell+4}} \Bigg\{ M I_L^{(\ell+1)} \mathcal{I}_L^{(\ell+1)} + 2M \sum_{p=1}^{\ell} (-)^p I_L^{(\ell+1-p)} \mathcal{I}_L^{(\ell+1+p)} + v^i \frac{\partial M}{\partial v^i} I_L^{(\ell+1)} \mathcal{I}_L^{(\ell+1)} - 2M(-)^\ell v^i \frac{\partial I_L}{\partial v^i} \mathcal{I}_L^{(2\ell+2)} + \delta E_\ell \Bigg\} + \left(I_L, \mathcal{I}_L, a_\ell\right) \to \left(J_L, \mathcal{J}_L, b_\ell/c^2\right),$$

$\hookrightarrow$ δE_ℓ is a non-trivial correction which logs contribute only for non-circular orbits[6], but yields a DC contribution that reads on average

$$\left\langle \sum_{\ell=2}^{\infty} \frac{G^2}{c^{2\ell+4}} \left(a_\ell\, \delta E_\ell + \frac{b_\ell}{c^2}\, \delta \tilde{E}_\ell \right) \right\rangle = -\frac{2GM}{c^3}\, \mathcal{F}_{\text{GW}}.$$

[6] It was originally discovered for $\ell = 2$ in L. Bernard *et al.*, PRD 95(2017)044026.

Contribution of the simple tails: circular orbits

- In the adiabatic approximation for quasi-circular orbits, $\mathcal{I}_L = -2\left(\log(r\omega/c) + \gamma_E\right) I_L$, so, in terms of the 1PN parameter $x \equiv (Gm\omega/c^3)^{2/3}$,

The simple tail's logs contribute as

$$E^{\log}_{\text{tail}} = -\frac{m\nu^2}{2} x^5 \log x \left[\frac{448}{15} - \left(\frac{4988}{35} + \frac{656}{5}\nu\right) x + \left(-\frac{1967284}{8505} + \frac{914782}{945}\nu + \frac{32384}{135}\nu^2\right) x^2 + \left(\frac{16785520373}{2338875} - \frac{1424384}{1575} \log\left(\frac{r}{r_0}\right) + \left(\frac{2131}{42}\pi^2 - \frac{41161601}{51030}\right)\nu - \frac{13476541}{5670}\nu^2 - \frac{289666}{1215}\nu^3\right) x^3 + \mathcal{O}\left(x^4\right) \right],$$

where the blue contributions were already known.[7]

(!) r_0 is the unphysical UV cutoff scale, it should disappear ! (!)

[7] see notably A. Le Tiec, L. Blanchet & B. Whiting, PRD 85(2012)064039, and D. Bini & T.Damour, PRD 89(2014)064063.

Contribution of the higher order tails

- The (tails)$^{n+1}$ arise at 1.5PN beyond the (tails)^{n}:

$\Rightarrow$ the tails-of-tails contribute at 5.5PN, 6.5PN...
and don't bear log dependencies, at least up to 7.5PN,[8]

$\Rightarrow$ the tails-of-tails-of-tails hit at 7PN, with log and log^2 contribution !
But no general action yet, contrary to the simple tails...

What we should do :

$\hookrightarrow$ Proceed as when proving the action in traditional PN's way, but iterating two times more the Einstein's equations...

$\rightsquigarrow$ work currently in progress
with L. Blanchet and T. Marchand.

What we did :

[8]L. Blanchet, G. Faye & B. Whiting, PRD 90(2014)044017.

Need to take a step back ? Let's redshift your mind !

- At leading order the (tails)^3 only contributes at $\nu^2 \Rightarrow$ test mass limit.

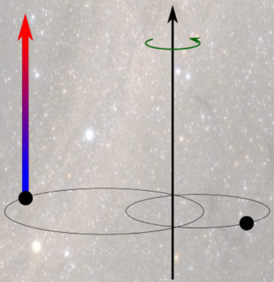
$\hookrightarrow$ Quantities in the test mass limit are known analytically up to 21.5PN,[9] can we use them ?

$\Rightarrow$ Yes ! Let's use the "Detweiler's redshift variable".[10]

$\hookrightarrow$ For a system with helicoidal symmetry (*ie.* with a Killing vector $K = \partial_t + \Omega\partial_\phi$).

$\hookrightarrow$ z defined as the redshift of a photon emitted by one of the masses, observed at infinity along the rotation axe.

$\hookrightarrow$ z^{-1} can also be seen as the invariant associated with K.



[9] S. Hooper, C. Kavanagh & A. C. Ottewill, PRD 93(2016)044010.

[10] S. Detweiler, PRD 77(2008)124026.

From redshift to conserved energy

- In the test mass limit, the redshift reads $z_1 = \sqrt{1-3x} + \nu z_{SF} + \ldots$

$\Rightarrow$ Using the first law of binary point-like particle mechanics

$$\delta M - \omega\,\delta J = z_1\,\delta m_1 + z_2\,\delta m_2,$$

one can express the energy as a function of z_{SF}.[11]
The logs contributions are contained in $\mathcal{E} = \nu\, z_{SF}/2 - \nu\, x\, z'_{SF}/3$.

$\Rightarrow$ With the results of [12], we can compute the contribution of (tails)^3.

The leading (tails)^3 contributes as

$$E^{\log}_{(\text{tails})^3} = -\frac{m\nu^2}{2}x^8 \log x\left[-\frac{356096}{1575}\log x - \frac{108649792}{55125} + \frac{1424384}{1575}\left(\log\left(\frac{r}{r_0}\right) - \gamma_E - \log 4\right)\right].$$

[11] A. Le Tiec, E. Barausse & A. Buonanno, PRL 108(2011)131103.
[12] S. Hooper, C. Kavanagh & A. C. Ottewill, PRD 93(2016)044010.

Computing the dominant log contribution

- In general, the $\log^n(x)$ arises at $(3n+1)$PN (*cf.* the $\log^2$ at 7PN).
- It is possible to compute their contributions *via* renormalization group techniques: the Bondi mass and angular momentum run as[13]

$$\mu \frac{\mathrm{d}M(\mu)}{\mathrm{d}\mu} = -\frac{2G^2M}{5}\left[2I^{(1)}_{ij}I^{(5)}_{ij} - 2I^{(2)}_{ij}I^{(4)}_{ij} + I^{(3)}_{ij}I^{(3)}_{ij}\right],$$

$$\mu \frac{\mathrm{d}J^i(\mu)}{\mathrm{d}\mu} = -\frac{8G^2M}{5}\epsilon^{ijk}\left[I_{jl}I^{(5)}_{kl} - I^{(1)}_{jl}I^{(3)}_{kl} + I^{(2)}_{jl}I^{(3)}_{kl}\right].$$

$\Rightarrow$ But the quadrupole scales following

$$I_{ij}(t,\mu) = \sum_{n=0}^{\infty}\frac{\left(-\beta_I G^2M^2\right)^n}{n!}\log^n\left(\frac{\mu}{\mu_0}\right) I^{(2n)}_{ij}(t,\mu_0), \qquad \beta_I = -\frac{214}{105},$$

thus one can integrate $M(\mu)$ and $J^i(\mu)$.

[13] W. D. Goldberger, A. Ross & I. Z. Rothstein, PRD 89(2014)124033.

Computing the dominant log contribution

- Averaging over one period and reducing in circular orbits, we obtain expressions for $E = M - m$ and J in terms of $\gamma = Gm/r$ and ω.

↪ But E and J are linked by the "thermodynamical" relation

$$\frac{\mathrm{d}E}{\mathrm{d}\omega} = \omega \, \frac{\mathrm{d}J}{\mathrm{d}\omega},$$

which allows to derived $\gamma(\omega)$.

⇒ Thus we can express $E(x)$ and $J(x)$ (where $x \equiv (G\omega/c^3)^{2/3}$):

$$E_{\text{dom logs}} = -\frac{m\nu x}{2}\left[\frac{64\nu}{15}\sum_{n=1}^{\infty}\frac{6n+1}{n!}\,(4\beta_I)^{n-1}\,x^{3n+1}\log^n x\right],$$

$$J_{\text{dom logs}} = -\frac{m^2\nu}{\sqrt{x}}\left[\frac{64\nu}{15}\sum_{n=1}^{\infty}\frac{3n+2}{n!}\,(4\beta_I)^{n-1}\,x^{3n+1}\log^n x\right].$$

Conclusion & final result

- Tails are interesting non-linear effects arising in GR, with a peculiar logarithmic behaviour.

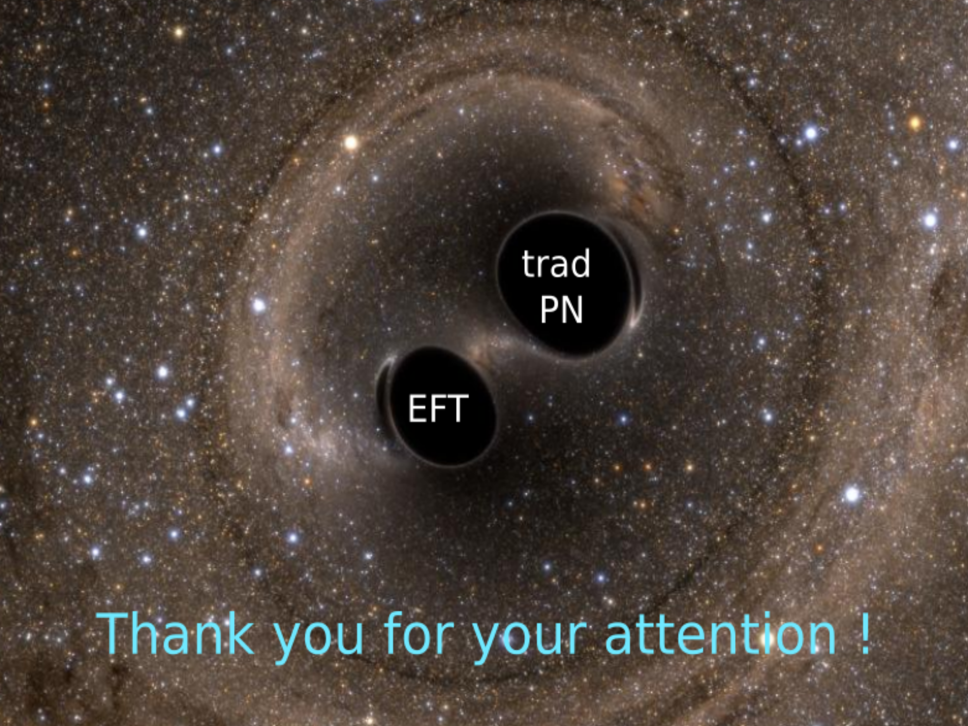
↪ We have derived the log contributions of simple tails in the energy,

↪ and, using different approaches, the leading (tails)^3 and $\log^n$ ones.

The logarithmic contributions of the tails read

$$E^{\log}_{\text{tails}} = -\frac{m\nu^2}{2}x^5 \log x \left[\frac{448}{15} - \left(\frac{4988}{35} + \frac{656}{5}\nu\right)x + \left(-\frac{1967284}{8505} + \frac{914782}{945}\nu + \frac{32384}{135}\nu^2\right)x^2 \right.$$
$$+ \left(\frac{85229654387}{16372125} - \frac{1424384}{1575}(\gamma_E + \log 4) + \left(\frac{2131}{42}\pi^2 - \frac{41161601}{51030}\right)\nu \right.$$
$$\left. - \frac{13476541}{5670}\nu^2 - \frac{289666}{1215}\nu^3 - \frac{35696}{1575}\log x\right)x^3$$
$$\left. + \sum_{n=3}^{\infty} \frac{64}{15}\frac{6n+1}{n!}(4\beta_I)^{n-1}x^{3n+1}\log^n x + \dots \right],$$

! But most of all, this work shows the great potential of synergies ! between traditional PN methods and EFT ones.


trad
PN
EFT
Thank you for your attention !