The 4PN phase of non-spinning compact binary systems: where are we ?

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Outline

1) Introduction & overview of the PN formalism

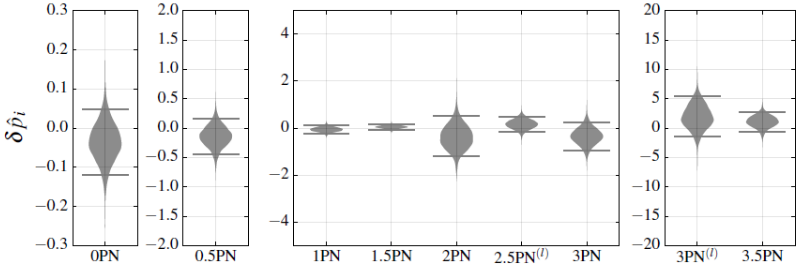
2) What has been done

3) What is left to do

4) Summary and conclusion

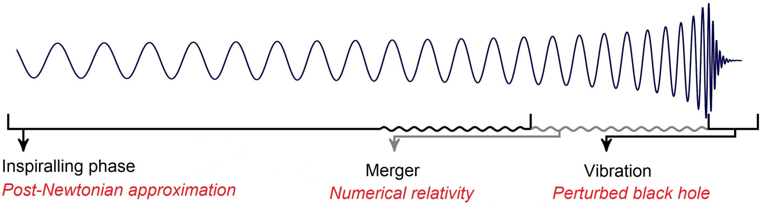
Motivations for the 4PN waveform

- Needed for future detectors (e.g. LISA).
- More accurate determination of the astrophysical parameters (masses, spins).
- Comparison with numerical relativity and self-force calculations.
- Better comparison between GR and alternative theories of gravity.



Combined posteriors for GW150914, GW151226 & GW170104.
Figure from *LSC* & *Virgo*, PRL **118**(2018)221101.

The post-Newtonian formalism



PN formalism:

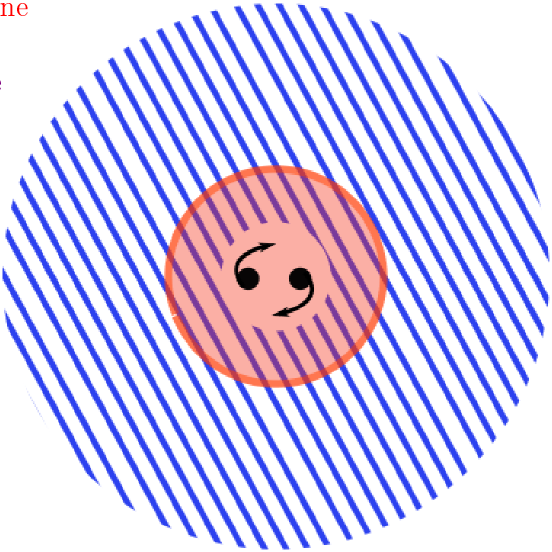
- Perturbative expansion of the equations of GR.

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \quad \text{with} \quad \tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}(h, \partial h, \partial^2 h)$$

- Weak field, small velocities : $(v/c) \ll 1$.
- 4^{th} PN order $\rightarrow O(1/c^8)$ beyond the quadrupole formula.

Different space zones in the PN formalism

Exterior zone
Far zone
Buffer zone



The GW phase : what is known ?

PN parameter : $x \equiv \left(\frac{GM\omega}{c^3}\right)^{2/3}$

Angular frequency : ω

$$\varphi = \int \omega \, \mathrm{d}t$$

$$\varphi = -\frac{1}{32\nu x^{5/2}} \Big[1 + \varphi_{1\mathrm{PN}} x + \varphi_{1.5\mathrm{PN}} x^{3/2} + \varphi_{2\mathrm{PN}} x^2 + \varphi_{2.5\mathrm{PN}} x^{5/2} + \varphi_{3\mathrm{PN}} x^3 + \varphi_{3.5\mathrm{PN}} x^{7/2} + \varphi_{4\mathrm{PN}} x^4 + O(x^{9/2}) \Big]$$

The GW phase : what is known ?

$$\varphi_{1\text{PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{1.5\text{PN}} = -10\pi$$

$$\varphi_{2\text{PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

$$\varphi_{2.5\text{PN}} = \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi \ln(x)$$

$$\varphi_{3\text{PN}} = \frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_{\text{E}} - \frac{3424}{21}\ln 2 + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3 - \frac{856}{21}\ln(x)$$

$$\varphi_{3.5\text{PN}} = \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi$$

Brief overview of the steps to compute the phase

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$$\sigma = \rho + \dots, \qquad V = -4\pi G \Box^{-1}\sigma = \frac{G m_1}{r_1} + \dots$$

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Computation of the potentials

Sources:

$$\sigma = \frac{T^{00} + T^{ii}}{c^2}, \qquad \sigma_i = \frac{T^{0i}}{c}, \qquad \sigma_{ij} = T^{ij}$$

For point particles $\sigma \propto \delta^{(3)}(\vec{x} - \vec{y}_A)$.

Potentials:

The potentials fully parametrize the metric.

$$\begin{aligned} \Box V &= -4\pi G \sigma \\ \Box \hat{W}_{ij} &= -4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V \partial_j V \end{aligned}$$

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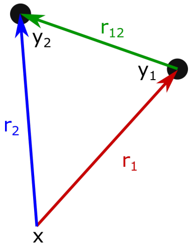
Defining $\partial_{1i} \equiv \partial/\partial y_1^i$,

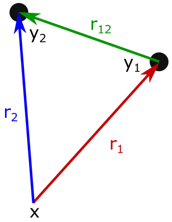
$$\partial_i V \partial_j V = G^2 m_1\, m_2\, \partial_{1i}\partial_{2j} \frac{1}{r_1 r_2} + \dots$$



$\hookrightarrow$ Need to know how to compute $\Box^{-1} \frac{1}{r_1 r_2}$

Computation of the kernels and matching

$$\Box \mathscr{G} = \frac{1}{r_1 r_2}, \qquad \Box F^{12} = \frac{r_1}{2r_2} \qquad \dots$$



1) Find a particular solution.

$$\mathscr{G} = g + \frac{1}{c^2}\partial_t^2 f + O\left(\frac{1}{c^4}\right)$$

$$\Delta g = \frac{1}{r_1 r_2} \qquad \Delta f = g$$

At Newtonian order, $g = \ln(r_1 + r_2 + r_{12})$.

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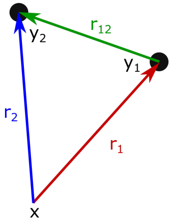

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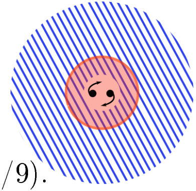
2) Match it to the far-zone in order to have $\mathscr{M}(P) = P$.
[Blanchet Living Review (2014)]



$\hookrightarrow$ Enables to compute some of the required potentials (5/9).
$\hookrightarrow$ Doesn't work for more complicated ones (no analytic formula).

Source multipole moments

$$\Sigma = \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} \qquad \Sigma_i = \frac{\overline{\tau}^{0i}}{c} \qquad \Sigma_{ij} = \overline{\tau}^{ij}$$

Σ, Σ_i and Σ_{ij} contain the σ, σ_i, σ_{ij} and the potentials $\{V, V_i, \hat{W}_{ij}, \dots\}$[1]

Multipole moments : [Blanchet Living Review (2014)]

$$\mathrm{I}_L = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3x \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \left[\delta_\ell \hat{x}_L \Sigma + \frac{\alpha_\ell}{c^2} \hat{x}_{iL} \dot{\Sigma}_i + \frac{\beta_\ell}{c^4} \hat{x}_{ijL} \ddot{\Sigma}_{ij}\right] \left(x, u + \frac{zr}{c}\right)$$

Similar expression for the current multipoles J_L.

$\hookrightarrow$ Some potentials are not known in all space $\Rightarrow$ IBP $\Rightarrow$ surface terms.

[1] $\tau^{\mu\nu} = |g| T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}$

The mass quadrupole

3 types of terms:[2]

- compact support: 168 terms $\qquad (\sigma \propto \delta^{(3)}(\vec{x} - \vec{y}_A))$

$$\int \mathrm{d}^3x \, \hat{x}^{ij} \sigma V$$

- non-compact support: 419 terms

$$\int \mathrm{d}^3x \, r^B \hat{x}^{ij} \, \hat{W}_{ab} \partial_{ab} V$$

- surface: 67 terms

$$\int \mathrm{d}^3x \, r^B \hat{x}^{ij} \Delta(V^2)$$

[2] For 4PN, 654 terms while "only" 92 for 3PN

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$$\partial_{ab} \left(\frac{1}{r_1}\right)\Bigg|_{\mathrm{Distr}} = -\frac{4\pi}{3} \delta_{ab}\, \delta^{(3)}(\vec{x} - \vec{y}_1)$$

- surface: 67 terms

$$\int \mathrm{d}^3x\, r^B \hat{x}^{ij} \Delta(V^2)$$

$\hookrightarrow$ Had to take into account distributional parts.

We **completed** the integration of the mass quadrupole with the Hadamard regularisation.

[2] For 4PN, 654 terms while "only" 92 for 3PN

The regularisation problems : UV and IR

$\hookrightarrow$ Formalism written using the Hadamard *Partie Finie* regularisation.

$$\mathrm{I}_L = \boxed{\underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3x \left(\frac{r}{r_0}\right)^B} \int_{-1}^{1} \mathrm{d}z \left[\delta_\ell \hat{x}_L \Sigma + \frac{\alpha_\ell}{c^2} \hat{x}_{iL} \dot{\Sigma}_i + \frac{\beta_\ell}{c^4} \hat{x}_{ijL} \ddot{\Sigma}_{ij}\right] \left(x, u + \frac{zr}{c}\right)$$

$\hookrightarrow$ Crucial to distinguish between UV (on the bodies) and IR (infinity) regularisations.

- UV : bodies modelled as point particles.
- IR : need for a regularisation at infinity in the formalism itself.

$\hookrightarrow$ But dimensional regularisation is more suitable at high order (3PN for UV and 4PN for IR)

$$\mathscr{D}I \equiv I^{(d)} - I^{(\mathrm{Had})}$$

[PRD **71**, 124004 (2005)]

$\hookrightarrow$ Now computing the UV differences on the mass quadrupole.

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Next steps to achieve the computation of the mass quadrupole

- Complete the computation of the UV regularised (in d-dim) I_{ij}.

- Compute the IR difference for the mass quadrupole.

- Reduce it into the CoM frame for circular orbits.

Next steps to achieve the computation of the 4PN phase

- Compute the other multipoles (using the same method), $3 \leqslant \ell \leqslant 6$

$$\mathrm{I}_L = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3x \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \left[\delta_\ell \hat{x}_L \Sigma + \frac{\alpha_\ell}{c^2} \hat{x}_{iL} \dot{\Sigma}_i + \frac{\beta_\ell}{c^4} \hat{x}_{ijL} \ddot{\Sigma}_{ij}\right] \left(x, u + \frac{zr}{c}\right).$$

- Compute the non-linear terms.

$$\mathrm{U}_{ij} = \mathrm{I}_{ij}^{(2)} + (\text{non linear terms})$$

- Compute the flux

$$\mathscr{F} = \frac{G}{c^5} \left[\frac{1}{5} \mathrm{U}_{ij}^{(1)} \mathrm{U}_{ij}^{(1)} + \frac{1}{c^2} \left(\frac{1}{189} \mathrm{U}_{ijk}^{(2)} \mathrm{U}_{ijk}^{(2)} + \frac{16}{45} \mathrm{V}_{ij}^{(1)} \mathrm{V}_{ij}^{(1)}\right) + \dots\right].$$

- Deduce the phase through the balance equation

$$\mathscr{F} = -\frac{\mathrm{d}E}{\mathrm{d}t} \qquad \Rightarrow \qquad \varphi = -\int \omega(x) \frac{\mathrm{d}E/\mathrm{d}x}{\mathscr{F}(x)} \mathrm{d}x.$$

Summary

What has been done:

- We computed the potentials required for the 4PN multipoles.
- We computed the mass quadrupole in 3d.
- Now completing the computation the UV (dim-reg/Had) difference.

What is left to do for the mass quadrupole:

- Compute the IR (dim-reg/Had) difference.
- Reduce the mass quadrupole in the CoM frame for circular orbits.

What is left to do for the 4PN flux:

- Compute the other multipoles (much easier).
- Compute non-linear terms.
- Reduce these quantities in the CoM frame for circular orbits.
- Compute the flux.
- Deduce the phase.